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Original Article

Generalized Topp-Leone family of distributions

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ABSTRACT

Background & Aim: Adding parameters to an existing distribution to expand the family of distributions is a very common approach for developing more flexible models. Several ways for generating new distributions from classic ones have been developed.

Methods & Materials: A generalization of Topp-Leone generator of distributions was introduced. Several of its fundamental properties were obtained such as quantiles, moments, moment generating function (MGF), order statistics and maximum likelihood estimator (MLE).

Results: We provided four sub-models of the new family which extended some of the basic lifetime models such as exponential, Weibull, gamma and generalized exponential distributions. These distributions exhibited a wide range of shapes varying skewness and different forms of hazard rate function (HRF).

Conclusion: We have provided four new distributions. The flexibility of the proposed distributions and increased range of skewness were able to fit and capture features in one real dataset much better than some competitor distributions.

Introduction

Generating family of distributions is a new improvement for creating and extending the usual classical distributions. Several ways for generating new distributions from classic ones have been developed. Hence, several classes to generate new distributions by adding one or more parameters have been proposed in the statistical literature. For example, Beta-G distribution family that was introduced by Eugene et al. (1), McDonald class of distributions by Alexander et al. (2), gamma-G type 1 by Zografos and Balakrishnan (3) and Amini et al. (4), gamma-G type 2 by Ristic and Balakrishnan (5), odd exponentiated generalized by Cordeiro et al. (6), transformed-transformer (T-X) by Alzaatreh et al. (7), exponentiated T-X by Alzaghal et al. (8), odd Weibull-G by Bourguignon et al. (9), exponentiated half-logistic by Cordeiro et al. (10), T-X{Y} quantile-based approach by Aljarrah et al. (11), T-R{Y} by Alzaatreh et al. (12), Lomax-G by Cordeiro et al. (13), Kumaraswamy-G class of distributions by Cordeiro et al. (14), Kumaraswamy odd log-logistic-G by Alizadeh et al. (15), logistic-X by Tahir et al. (16) and alpha power transformation family of distributions introduced by Mahdavi and Kundu (17). Recently, Topp-Leone generated (TL-G) family of distributions proposed by Al-Shomrani et al. (18) with its cumulative distribution function (CDF) and probability density function (PDF) as follows:

$$F_{TL\text{-}G}(x,\alpha,\xi) = [G(x,\xi)]^{\alpha}\,[2 - G(x,\xi)]^{\alpha},\ \alpha > 0,$$

and

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$f_{TL\text{-}G}(x,\alpha,\xi) = 2\alpha g(x,\xi)[G(x,\xi)]^{\alpha-1}\,[2 - G(x,\xi)]^{\alpha-1}\,[1 - G(x,\xi)],\ \alpha > 0,$

where G(x,ξ) and g(x,ξ) are the CDF and PDF of a baseline continuous distribution with parameter vector ξ.

The aim of this paper was to introduce an extra parameter to the TL-G family of distributions to bring more flexibility. We called this new generator as generalized Topp-Leone generated (GTL-G) family of distributions. We used GTL-G method to exponential, Weibull, gamma and generalized exponential distributions and generated four new families of distributions. It was observed that the new distributions had several desirable properties and they could be used as an alternative to the several lifetime models.

Methods

Let G(x,ξ) be the CDF of a continuous random variable X then the CDF of GTL-G family is defined as

$$F_{GTL\text{-}G}(x,\alpha,\beta,\xi) = \frac{[G(x,\xi)]^{\alpha}\,[\beta - G(x,\xi)]^{\alpha}}{(\beta-1)^{\alpha}} \qquad (1)$$

α > 0, β ≥ 2, where ξ is the baseline parameter vector.

The GTL-G family reduces to the baseline distribution G(x,ξ) if

$$\alpha = \frac{\log G(x,\xi)}{\log\{G(x,\xi)(\beta-G)(x,\xi))\} - \log(\beta-1)}$$

By differentiating of (1) we get the PDF as

$$f_{GTL\text{-}G}(x,\alpha,\beta,\xi) = \frac{\alpha}{(\beta-1)^{\alpha}}\, g(x,\xi)[G(x,\xi)]^{\alpha-1}\,[\beta - G(x,\xi)]^{\alpha-1}\,[\beta - 2G(x)],$$

where g(x,ξ) is the PDF of random variable X It is clear that for fixed β = 2 the GTL-G family reduces to the TL-G family of distributions. The survival function and the hazard rate function (HRF) for GTL-G family of distributions are in the following forms, respectively:

$$S_{GTL\text{-}G}(x,\alpha,\beta,\xi) = 1 - \frac{[G(x,\xi)]^{\alpha}[\beta-G(x,\xi)]^{\alpha}}{(\beta-1)^{\alpha}},$$

and

$$h_{GTL-G}(x,\alpha,\beta,\xi) = \frac{\alpha g(x,\xi)[G(x,\xi)]^{\alpha-1}}{(\beta-1)^{\alpha} - [G(x,\xi)]^{\alpha-1}}\,\frac{[\beta - G(x,\xi)]^{\alpha-1}[\beta-2G(x)]}{[\beta-2G(x)]}.$$

The p-th quantile function of GTL-G family is given by

$$Q_{GTL-G}(p,\alpha,\beta,\xi) = Q_G\left(\frac{\beta-\sqrt{\beta^2-4(\beta-1)p^{\frac{1}{\alpha}}}}{2},\xi\right),$$

where $Q_G(.,\xi)$ is the quantile function of the baseline distribution.

The properties and inferences about GTL-G family can be easily obtained from the following expansion for the PDF and CDF of GTL-G family. Considering the series representations

$$(\beta - x)^{\alpha} = \sum_{j=0}^{\infty} C(\alpha, j)(-1)^j \beta^{\alpha-j} x^j \qquad (2)$$

where

$$C(\alpha, j) = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-j+1)}{j!}$$

$$\text{and } \left|\frac{x}{\beta}\right| \le 1,$$

Hence,

$$F_{GTL-G}(x,\alpha,\xi) = \sum_{j=0}^{\infty}\frac{1}{(\beta-1)^{\alpha}}C(\alpha,j)\,(-1)^j\beta^{\alpha-j}[G(x,\xi)]^{\alpha+j},$$

and

$$f_{GTL-G}(x,\alpha,\xi) = g(x,\xi)\sum_{j=0}^{\infty}\frac{\alpha+j}{(\beta-1)^{\alpha}}\,C(\alpha,j)(-1)^j\beta^{\alpha-j}[G(x,\xi)]^{\alpha+j-1} \qquad (3)$$

The moment of the GTL-G family can be computed by probability weighted moments of order (s,r) of the baseline distribution. The probability weighted moments of order (s,r) is defined by Greenwood et al. (19) as:

$$\tau_{s,r} = E\left(Y^s[G(Y,\xi)]^r\right) = \int_{-\infty}^{\infty} y^s[G(y,\xi)]^r g(y,\xi)dy.$$

Therefore, using (3) the s-th moment of GTL-G family can be derived by

$$E(X^s)=\sum_{j=0}^{\infty}\frac{\alpha+j}{(\beta-1)^{\alpha}}C(\alpha,j)(-1)^j\beta^{\alpha-j}\tau_{s,\alpha+j-1}.$$

The moment generating function (MGF) can be obtained of the following series expansion

$$M_X(t,\alpha,\xi)=E(e^{tX})=\sum_{s=0}^{\infty}\frac{t^sE(X^s)}{s!}.$$

Or using expansion for the density in (3) the MGF can be obtained as:

$$M_X(t,\alpha,\xi)=\sum_{j=0}^{\infty}\frac{\alpha+j}{(\beta-1)^{\alpha}}C(\alpha,j)(-1)^j\int_0^1 e^{tQ_G(u,\xi)}u^{\alpha+j-1}du.$$

Let $X_1,X_2,\ldots,X_n$ be a random sample from the GTL-G family. Denote the random variables in the ascending order by $Y_1\le Y_2\le\ldots\le Y_n$ The PDF of Y_i can be express as:

$$f_{Y_i}(y)=\frac{1}{B(i,n-i+1)}f_{GTL-G}(y)[F_{GTL-G}(y)]^{i-1}[1-F_{GTL-G}(y)]^{n-i},$$

where $B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ is the beta function. Using binomial expansion for $[1-F_{GTL-G}(y)]^{n-i}$ we got:

$$f_{Y_i}(y)=\frac{1}{B(i,n-i+1)}f_{GTL-G}(y)\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k[F_{GTL-G}(y)]^{i+k-1} \quad (4)$$

Referring to (1), $[F^{GTL-G}(y)]^{i+k-1}$ becomes

$$[F_{GTL-G}(y)]^{i+k-1}=\frac{[G(y)]^{\alpha(i+k-1)}}{(\beta-1)^{\alpha(i+k-1)}}[\beta-G(y)]^{\alpha(i+k-1)} \quad (5)$$

Using series expansion defined in (2) equation (5) becomes

$$[F_{GTL-G}(y)]^{i+k-1}=\frac{1}{(\beta-1)^{\alpha(i+k-1)}}\sum_{j=0}^{\infty}(-1)^j\beta^{\alpha(i+k-1)-j}C(\alpha(i+k-1),j)[G(y)]^{\alpha(i+k-1)+j} \quad (6)$$

Substituting (6) in (4), we got

$$f_{Y_i}(y)=\frac{1}{B(i,n-i+1)}f_{GTL-G}(y)\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}}{(\beta-1)^{\alpha(i+k-1)}}\sum_{j=0}^{\infty}(-1)^{k+j}\beta^{\alpha(i+k-1)-j}C(\alpha(i+k-1),j)[G(y)]^{\alpha(i+k-1)+j} \quad (7)$$

Now, by substituting the obtained expansion for f_{GTL-G} defined by (3) in (7), the PDF for Y_i can be expressed as CDF and PDF of baseline distribution by

$$f_{Y_i}(y)=\frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}}{(\beta-1)^{\alpha(i+k)}}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+j+l}(\alpha+l)\beta^{\alpha(i+k)-j-l}C(\alpha,l)C(\alpha(i+k-1),j)[G(y)]^{\alpha(i+k)+j+l-1}.$$

Results

***Special Models**:* In this section, we provided four special cases of the GTL-G family. These special models generalized some well-known distributions in the literature of lifetime analysis. The four baseline models used to generate special models were exponential, Weibull, gamma and generalized exponential distribution introduced by Gupta and Kundu (20).

***GTL exponential (GTL-E) distribution**:* We introduce the GTL-E distribution by taking $G(x,\xi)$ in (1) as CDF of an exponential distribution with mean $1/\lambda$.

***Definition 1**:* The non-negative random variable X has the GTL-E distribution denoted by $GTLE(\alpha,\beta,\lambda)$ with the shape parameters as $\alpha\neq 0$ and $\beta\geq 2$ and scale parameter $\lambda>0$ if the CDF of X is

$$F_{GTLE}(x,\alpha,\beta,\lambda)=\frac{(1-e^{-\lambda x})^{\alpha}(\beta-1+e^{-\lambda x})^{\alpha}}{(\beta-1)^{\alpha}},$$

the PDF of GTL-E distribution is given by

$$f_{GTLE}(x,\alpha,\beta,\lambda)=\frac{\alpha\lambda}{(\beta-1)^{\alpha}}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}(\beta-1+e^{-\lambda x})^{\alpha-1}(\beta-2+2e^{-\lambda x}).$$

Figure 1 (a) shows some of the different shapes of GTLE(α,β,λ) for selected values of the shape parameters and fixed scale parameter $\lambda = 1$. It is a unimodal and right-skewed function if $\alpha > 1$ and a decreasing function if $\alpha \leq 1$.

The survival function and HRF for the GTL-E distribution are given in the following forms:

$$S_{GTLE}(x,\alpha,\beta,\lambda)=1-\frac{(1-e^{-\lambda x})^{\alpha}(\beta-1+e^{-\lambda x})^{\alpha}}{(\beta-1)^{\alpha}},$$

and

$$h_{GTLE}(x,\alpha,\beta,\lambda)=\frac{\alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}}{(\beta-1)^{\alpha}-(1-e^{-\lambda x})^{\alpha}(\beta-1+e^{-\lambda x})^{\alpha}}(\beta-1+e^{-\lambda x})^{\alpha-1}(\beta-2+2e^{-\lambda x}).$$

Different shapes of the HRF are plotted in figure 2 (a). It is depicted in figure 2 (a) that the HRF of the GTL-E distribution can take decreasing shape if $\alpha \leq 1$ and upside-down bathtub or increasing shape for $\alpha > 1$ both converge to λ for $x \to \infty$.

The *p*-th quantile function of GTL-E distribution is given by

$$x_{GTLE}(p,\alpha,\beta,\lambda)=\frac{-1}{\lambda}\log\left(1-\frac{\beta-\sqrt{\beta^{2}-4(\beta-1)p^{\frac{1}{\alpha}}}}{2}\right).$$

It is easy to generate random numbers from GTLE distribution by using the following simple formula:

$$X=\frac{-1}{\lambda}\log\left(1-\frac{\beta-\sqrt{\beta^{2}-4(\beta-1)U^{\frac{1}{\alpha}}}}{2}\right),$$

where U is a uniformly distributed random variable on (0, 1) interval.

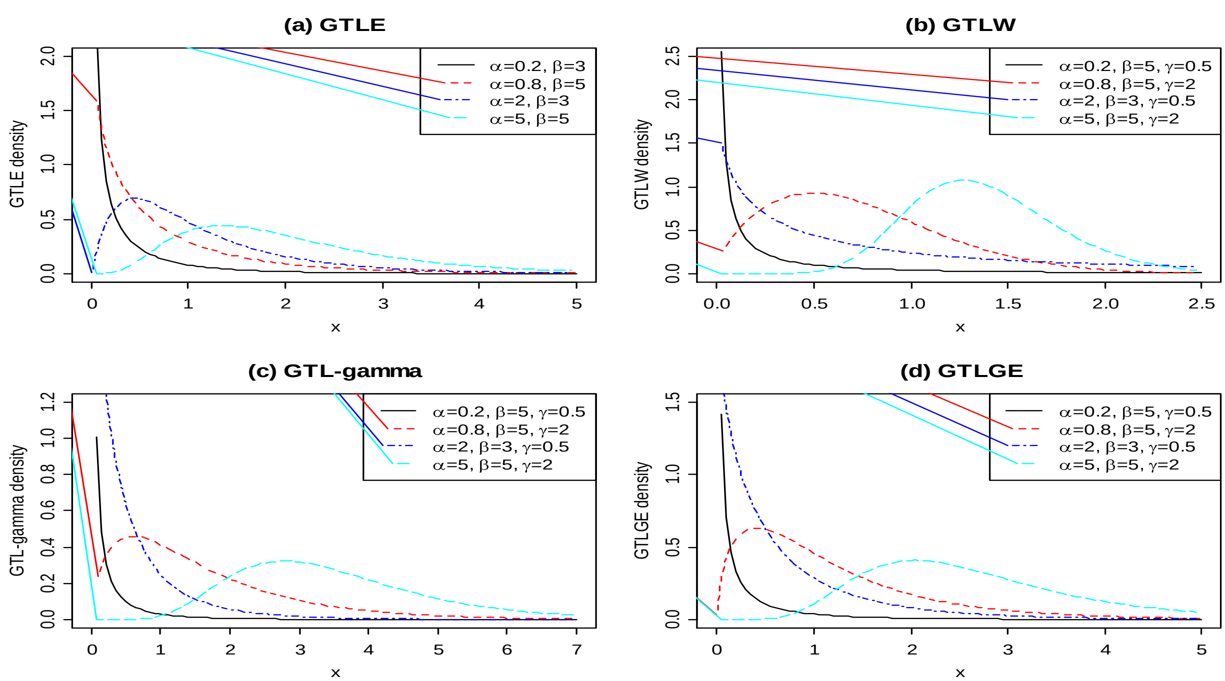

Figure 1 (a-d): The probability density function (PDF) plots for generalized Topp-Leone generated (GTL-G) sub-models with various shape parameters and fixed scale parameter $\lambda = 1$.
GTL-E: Generalized Topp-Leone exponential; GTL-W: Generalized Topp-Leone Weibull; GTL-gamma: Generalized Topp-Leone gamma; GTL-GE: Generalized Topp-Leone generalized exponential

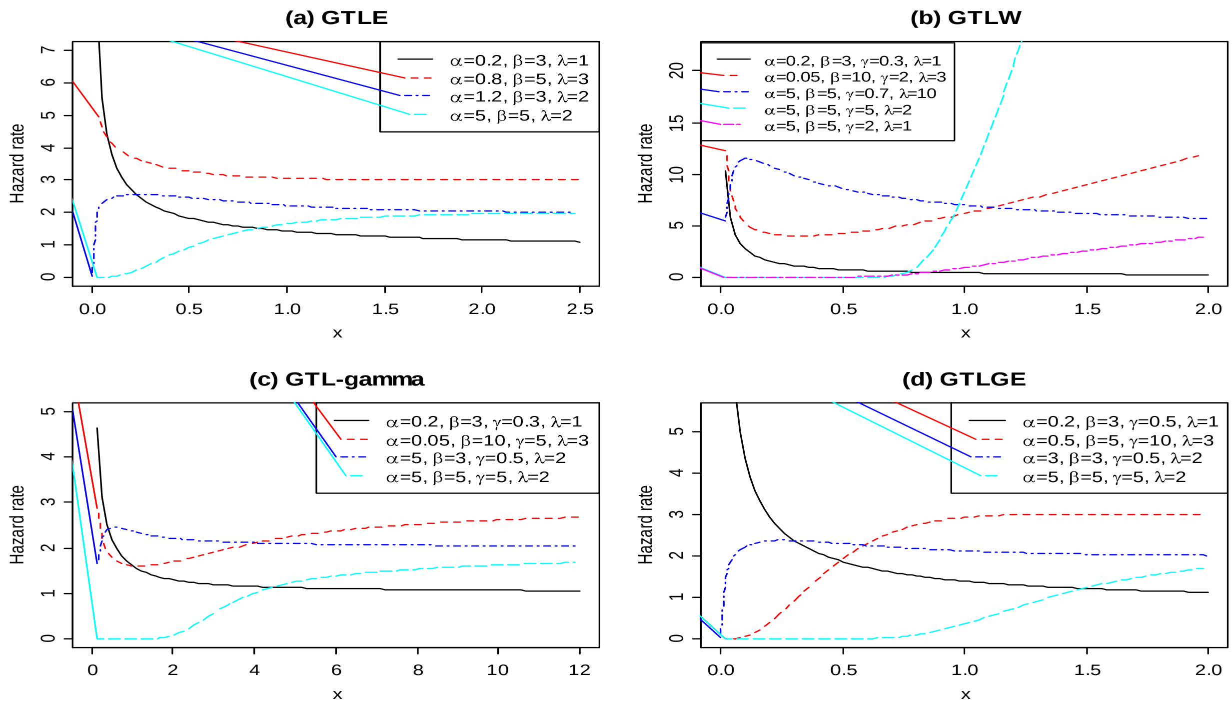

Figure 2 (a-d): The hazard rate function (HRF) plots for generalized Topp-Leone generated (GTL-G) sub-models with various parameters

GTL-E: Generalized Topp-Leone exponential; GTL-W: Generalized Topp-Leone Weibull; GTL-gamma: Generalized Topp-Leone gamma; GTL-GE: Generalized Topp-Leone generalized exponential

***GTL Weibull (GTL-W) distribution**:* GTL-W distribution is defined by taking G(x,ξ) in (1) as a CDF of Weibull distribution with the following CDF:

$$G(x,\gamma,\lambda) = 1 - e^{-\lambda x^{\gamma}},\ x > 0,\ \lambda > 0,\ \lambda > 0.$$

Definition 2: The non-negative random variable X has the GTL-W distribution denoted by GTLW(α,β,γ,λ) with the shape parameters as $\alpha \neq 0$, $\beta \geq 2$ and $\gamma > 0$ and scale parameter $\lambda > 0$ if the CDF of X is

$$F_{GTLW}(x,\alpha,\beta,\gamma,\lambda) = \frac{(1-e^{-\lambda x^{\gamma}})^{\alpha}(\beta-1+e^{-\lambda x^{\gamma}})^{\alpha}}{(\beta-1)^{\alpha}} \qquad (8)$$

The corresponding PDF to (8) is

$$f_{GTLW}(x,\alpha,\beta,\gamma,\lambda) = \frac{\alpha\lambda\gamma x^{\gamma-1}}{(\beta-1)^{\alpha}} e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\alpha-1} (\beta-1+e^{-\lambda x^{\gamma}})^{\alpha-1}(\beta-2+2e^{-\lambda x^{\gamma}}).$$

Different shapes of GTLW (α, β, γ, λ) for selected values of the shape parameters and fixed scale parameter $\lambda = 1$ are plotted in figure 1 (b). It could be a unimodal function or a decreasing function depending on different values of shape parameters.

The survival function and HRF of a GTL-W distributed random variable are given by

$$S_{GTLW}(x,\alpha,\beta,\gamma,\lambda) = 1 - \frac{(1-e^{-\lambda x^{\gamma}})^{\alpha}(\beta-1+e^{-\lambda x^{\gamma}})^{\alpha}}{(\beta-1)^{\alpha}}$$

and

$$h_{GTLGE}(x,\alpha,\beta,\gamma,\lambda) = \frac{\alpha\gamma\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha\gamma-1}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha-1}}{(\beta-1)^{\alpha}-(1-e^{-\lambda x})^{\alpha\gamma}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha}} [\beta-2(1-e^{-\lambda x})^{\gamma}].$$

Various shapes of the HRF are plotted in figure 2 (b). This family can produce flexible hazard rate shapes such as decreasing, increasing, bathtub, upside-down bathtub and J shape. This fact implies that the GTL-W family can be very useful to fit different data sets with various shapes.

The p-th quantile function of GTLW is

$$x_{GTLW}(p,\alpha,\beta,\gamma,\lambda)=$$
$$\{\frac{-1}{\lambda}\log\Big(1-\frac{\beta-\sqrt{\beta^2-4(\beta-1)p^{\frac{1}{\alpha}}}}{2}\Big)\}^{\frac{1}{\gamma}}.$$

Generating random numbers from GTLW distribution could be easily accomplished using the following formula:

$$X=\{\frac{-1}{\lambda}\log\Big(1-\frac{\beta-\sqrt{\beta^2-4(\beta-1)U^{\frac{1}{\alpha}}}}{2}\Big)\}^{\frac{1}{\gamma}}$$

where U is a uniformly distributed random variable on (0, 1) interval.

GTL-gamma distribution*:* The gamma CDF with shape parameter $\gamma > 0$ and scale parameter $\lambda > 0$ is $G(x,\gamma,\lambda) = \Gamma_{\gamma x}(\gamma)/\Gamma(\gamma)$, $x > 0$, where $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t}\, dt$ gamma function is and $\Gamma_{\gamma x}(\gamma) = = \int_0^x t^{\gamma-1} e^{-t}\, dt$ is the incomplete gamma function. The GTL-gamma distribution by taking $G(x,\xi)$ in (1) as CDF of the gamma distribution is defined as follows:

Definition 3: The non-negative random variable X has the GTL-gamma distribution denoted by GTL-gamma($\alpha,\beta,\gamma,\lambda$) with the shape parameters as $\alpha \neq 0$, $\beta > 0$ and γ and scale parameter $\lambda > 0$ if the CDF of X is

$$F_{GTLG}(x,\alpha,\beta,\gamma,\lambda)=\frac{\left[\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}\left[\beta-\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}}{(\beta-1)^{\alpha}}.$$

The associated PDF reduces to

$$f_{GTLG}(x,\alpha,\beta,\gamma,\lambda)=\frac{\alpha\lambda^{\gamma}x^{\gamma-1}e^{-\lambda x}}{\Gamma(\gamma)(\beta-1)^{\alpha}}\left[\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha-1}\left[\beta-\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha-1}\left[\beta-2\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]. \quad (9)$$

Plots of the density function (9) for selected parameter values are given in figure 1 (c). It could be a unimodal function or a decreasing function depending on values of the shape parameters.

The survival function and HRF for the GTL-gamma distribution are given in the following forms

$$S_{GTLG}(x,\alpha,\beta,\gamma,\lambda)=1-\frac{\left[\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}\left[\beta-\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}}{(\beta-1)^{\alpha}},$$

and

$$h_{GTLG}(x,\alpha,\beta,\gamma,\lambda)=\alpha\lambda^{\gamma}x^{\gamma-1}e^{-\lambda x}\frac{\left[\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha-1}\left[\beta-\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha-1}\left[\beta-2\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]}{\Gamma(\gamma)\left\{(\beta-1)^{\alpha}-\left[\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}\left[\beta-\frac{\Gamma_{\lambda x}(\gamma)}{\Gamma(\gamma)}\right]^{\alpha}\right\}}.$$

The plots of the HRF are plotted in figure 2 (c). It can take decreasing, increasing and upside-down bathtub shape depending on values of the shape parameters. Both converge to λ for $x \rightarrow \infty$.

GTL generalized exponential (GTL-GE) distribution*:* The CDF of generalized exponential distribution with parameters γ and λ is $G(x,\gamma,\lambda) = (1 - e^{-\lambda x})^{\gamma}$, $x > 0$. We introduced GTL-GE distribution by taking $G(x,\xi_2)$ in (1) as CDF of a generalized exponential distribution.

Definition 4: The non-negative random variable X has the GTL-GE distribution denoted by GTLGE($\alpha,\beta,\gamma,\lambda$), with the shape parameters as $\alpha \neq 0$, $\beta > 0$, and $\gamma > 0$ and scale parameter $\lambda > 0$ if the CDF of X is

$$F_{GTLGE}(x,\alpha,\beta,\gamma,\lambda)=\frac{(1-e^{-\lambda x})^{\alpha\gamma}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha}}{(\beta-1)^{\alpha}}.$$

The PDF of GTL-GE distribution reduces to

$$f_{GTLGE}(x,\alpha,\beta,\gamma,\lambda)=\frac{\alpha\gamma\lambda}{(\beta-1)^{\alpha}}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha\gamma-1}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha-1}[\beta-2(1-e^{-\lambda x})^{\gamma}].$$

The graph of the GTL-GE density in figure 1 (d) shows that the function could be unimodal or decreasing for the various values of the shapes parameters.

The survival function and HRF for the GTL-GE distribution are given in the following forms

$$S_{GTLGE}(x,\alpha,\beta,\gamma,\lambda)=1-\frac{(1-e^{-\lambda x})^{\alpha\gamma}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha}}{(\beta-1)^{\alpha}},$$

and

$$h_{GTLGE}(x,\alpha,\beta,\gamma,\lambda)=\frac{\alpha\gamma\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha\gamma-1}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha-1}}{(\beta-1)^{\alpha}-(1-e^{-\lambda x})^{\alpha\gamma}[\beta-(1-e^{-\lambda x})^{\gamma}]^{\alpha}}[\beta-2(1-e^{-\lambda x})^{\gamma}].$$

Plots of the HRF for GTL-GE distribution are displayed in figure 2 (d). It is depicted by figure 2 (d) that it can take decreasing, increasing and upside-down bathtub shape. Both converge to λ for $x\to\infty$.

The *p*-th quantile function of GTL-GE distribution is given by

$$x_{GTLGE}(p,\alpha,\beta,\gamma,\lambda)=\frac{-1}{\lambda}\log\left\{1-\left(\frac{\beta-\sqrt{\beta^2-4(\beta-1)p^{\frac{1}{\alpha}}}}{2}\right)^{\frac{1}{\gamma}}\right\}.$$

It is easy to generate random numbers from GTL-GE distribution by using the following simple formula:

$$X=\frac{-1}{\lambda}\log\left\{1-\left(\frac{\beta-\sqrt{\beta^2-4(\beta-1)U^{\frac{1}{\alpha}}}}{2}\right)^{\frac{1}{\gamma}}\right\},$$

where U is a uniformly distributed random variable on (0, 1) interval.

***Application: Guinea Pigs Dataset**:* In this section, we used an uncensored dataset corresponding to survival times of 72 guinea pigs injected with different amount of tubercle and was studied by Bjerkedal (21). The data represented the survival times of Guinea pigs in days. The data are given as below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

Many authors have been used this dataset to investigate their proposed models. For example, Gupta and Kundu (22) proposed weighted exponential distribution (WE) and compared it with Weibull, generalized exponential, and gamma distributions based on this dataset. They observed that the WE was better than all of them. Kharazmi et al. (23) proposed generalized WE distribution (GWE) and have been shown that the GWE has a better fit than WE for this dataset.

Here, we compared the four proposed sub-models of TL-G family with TL-GE distribution introduced by Sangsanit and Bodhisuwan (24) with PDF:

$$f(x)=2\alpha\beta\lambda e^{-\lambda x}\left[1-\left(1-e^{-\lambda x}\right)^{\beta}\right]\left(1-e^{-\lambda x}\right)^{\beta\alpha-1}\left[2-\left(1-e^{-\lambda x}\right)^{\beta}\right]^{\alpha-1},$$

TL-gamma distribution proposed by Rezaei et al. (25) with PDF:

$$f(x)=\frac{2\alpha\theta b^{a}x^{a-1}e^{-bx}[\Gamma_{bx}(a)]^{\theta\alpha-1}}{[\Gamma(a)]^{\theta a}}\left(1-\left[\frac{\Gamma_{bx}(a)}{\Gamma(a)}\right]^{\theta}\right)\left(2-\left[\frac{\Gamma_{bx}(a)}{\Gamma(a)}\right]^{\theta}\right)^{\alpha-1},$$

WE distribution with PDF

$$f(x,\alpha,\lambda)=\frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1-e^{-\alpha\lambda x}),$$

GWE distribution with PDF

$$f(x,\alpha,\lambda,n)=\frac{\alpha}{\beta(1/\alpha,n+1)}\lambda e^{-\lambda x}(1-e^{-\lambda\alpha x})^{n}.$$

Gamma exponentiated-exponential (GEE) distribution Ristic and Balakrishnan (5) with PDF

$$f(x,\alpha,\theta,\lambda)=\frac{\alpha\theta}{\Gamma(\lambda)}(1-e^{-\theta x})^{\alpha-1}[-\alpha\log(1-e^{-\theta x})]^{\lambda-1},$$

where $B(a,b) = \int_0^1 t^{a-1}(1 - t)^{b-1}dt$ is the beta function and $\alpha > 0, \beta > 0, \theta > 0, \lambda > 0$, n ∈ □, $x > 0$.

To see which one of these models was more appropriate to fit data the maximum likelihood estimation (MLE) of parameters, Akaike Information criterion (AIC) value, Bayesian Information Criterion (BIC) value, Kolmogorov-Smirnov (K-S) statistic and its associated P were obtained. The result is given in table 1. The GTL-E distribution gave the smallest AIC and smallest BIC. The smallest K-S statistic and the largest P belonged to GTL-G. In addition, the GTL-W and GTL-GE provided better fit to the data respect to competitor models. The histogram of the data and the plots of the fitted PDFs for the proposed special models are shown in figure 3. The probability plots for these special models are plotted in figure 4.

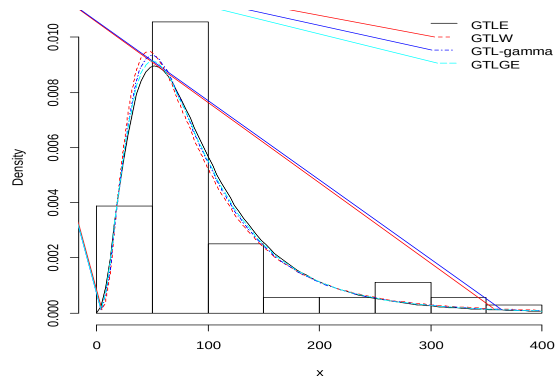

Figure 3. Fitted probability density function (PDF) for the Topp-Leone (TL) models to the guinea pigs data
GTL-E: Generalized Topp-Leone exponential; GTL-W: Generalized Topp-Leone Weibull; GTL-gamma: Generalized Topp-Leone gamma; GTL-GE: Generalized Topp-Leone generalized exponential

Discussion

We now determine the MLE values of the parameters of the GTL-G family of distributions from complete samples only.

Let $x_1,x_2,\ldots,x_n$ be a random sample from GTL-G distributions with unknown parameters $\Theta = (\alpha,\beta,\xi)^T$, a k × 1 parameter vector. Then, the log-likelihood function based on the given random sample is

$$\ell(\Theta) = n\log\alpha - n\alpha\log(\beta - 1) + \sum_{i=1}^{n}\log g(x_i,\xi) + (\alpha-1)\sum_{i=1}^{n}\log G(x_i,\xi) + (\alpha-1) + (\alpha - 1)\sum_{i=1}^{n}\log\{\beta - G(x_i,\xi)\} + \sum_{i=1}^{n}\log\{\beta - 2G(x_i,\xi)\}$$

The first order derivatives of $\ell(\Theta)$ are

$$\frac{\partial\ell(\mathbf{\Theta})}{\partial\alpha} = \frac{n}{\alpha} - n\log(\beta-1) + \sum_{i=1}^{n}\log G(x_i,\xi) + \sum_{i=1}^{n}\log\{\beta - 2G(x_i,\xi)\} = 0,$$

Table 1. The maximum likelihood estimation (MLE), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov test for fitted distributions

Model	MLE of the parameters	AIC	BIC	Kolmogorov-Smirnov statistics	P-value*
L-E	$\hat{\alpha}$ = 3.3369, $\hat{\beta}$ = 2.4267, $\hat{\lambda}$ = 0.0134	787.7422	794.5722	0.1052	0.403
GTL-W	$\hat{\alpha}$ = 15.8613, $\hat{\beta}$ = 8.6001, $\hat{\gamma}$ = 0.5248, $\hat{\lambda}$ = 0.3095	788.3582	797.4649	0.0956	0.526
GTL-gamma	$\hat{\alpha}$ = 10.1869, $\hat{\beta}$ = 2.2339, $\hat{\gamma}$ = 0.4693, $\hat{\lambda}$ = 0.0104	788.3149	797.4216	0.0932	0.559
GTL-GE	$\hat{\alpha}$ = 8.9250, $\hat{\beta}$ = 2.2328, $\hat{\gamma}$ = 0.4866, $\hat{\lambda}$ = 0.0123	788.8268	797.9335	0.0987	0.485
TL-GE	$\hat{\alpha}$ = 9.3376, $\hat{\beta}$ = 0.3910, $\hat{\lambda}$ = 0.0074	789.6965	796.5265	0.1164	0.284
TL-G	$\hat{\alpha}$ = 1.4630, $\hat{b}$ = 0.0076, $\hat{\alpha}$ = 57.1087, $\hat{\theta}$ = 0.0872	790.3751	799.4818	0.1125	0.322
WE	$\hat{\alpha}$ = 1.6232, $\hat{\lambda}$ = 0.0138	791.1382	795.6915	0.1173	0.275
GWE	$\hat{\alpha}$ = 4.4471, $\hat{\lambda}$ = 0.0141, n = 4	791.1382	794.3578	0.1098	0.350
GEE	$\hat{\alpha}$ = 2.6006, $\hat{\theta}$ = 0.0083, $\hat{\lambda}$ = 2.1138	793.2470	800.077	0.1347	0.147

* Kolmogorov-Smirnov test
MLE: Maximum likelihood estimation; GTL-E: Generalized Topp-Leone exponential; GTL-W: Generalized Topp-Leone Weibull; GTL-gamma: Generalized Topp-Leone gamma; GTL-GE: Generalized Topp-Leone generalized exponential; TL-GE: Topp-Leone generalized exponential; TL-G: Topp-Leone generalized; WE: Weighted exponential; GWE: generalized weighted exponential; GEE: Gamma exponentiated-exponential

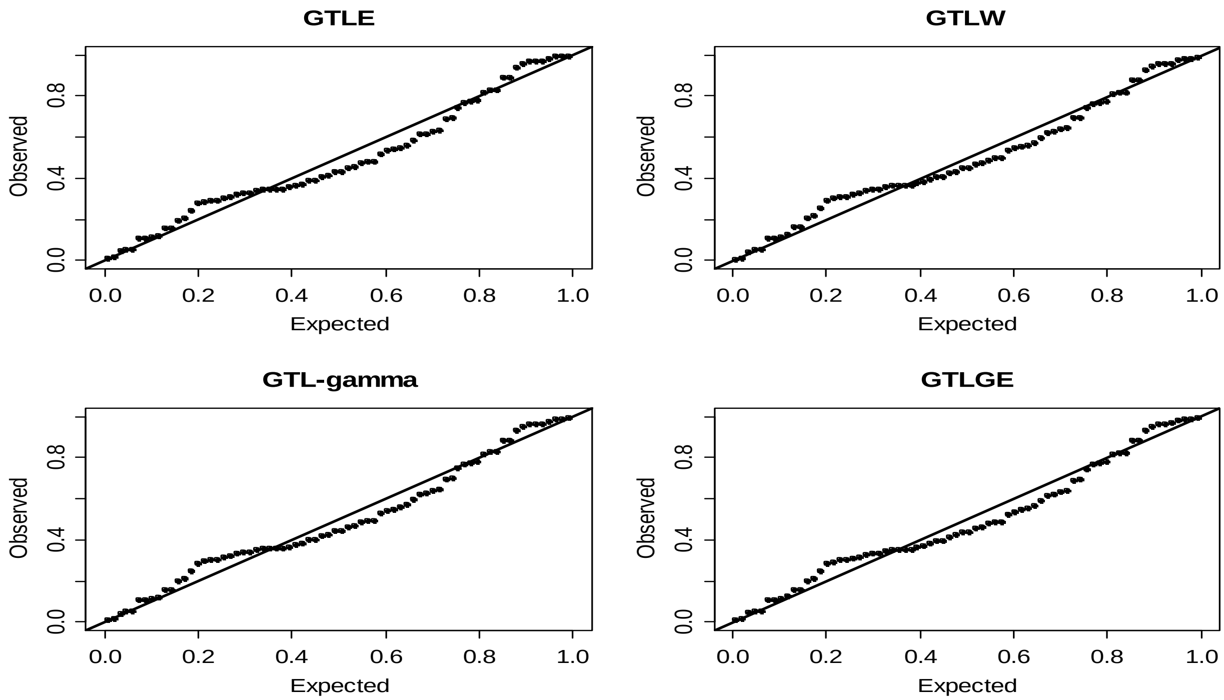

Figure 4. Probability plots for the generalized Topp-Leone (GTL) models
GTL-E: Generalized Topp-Leone exponential; GTL-W: Generalized Topp-Leone Weibull; GTL-gamma: Generalized Topp-Leone gamma; GTL-GE: Generalized Topp-Leone generalized exponential

$$\frac{\partial \ell(\boldsymbol{\Theta})}{\partial \beta} = \frac{-n\alpha}{\beta - 1} + (\alpha - 1)\sum_{i=1}^{n} \frac{1}{\beta - G(x_i, \xi)} + \sum_{i=1}^{n} \frac{1}{\beta - 2G(x_i, \xi)} = 0,$$

$$\frac{\partial \ell(\boldsymbol{\Theta})}{\partial \boldsymbol{\xi}} = \sum_{i=1}^{n} \frac{g^{(\xi)}(x_i, \xi)}{g(x_i, \xi)} + (\alpha - 1)\sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \xi)}{G(x_i, \xi)} - (\alpha - 1)\sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \xi)}{\beta - G(x_i, \xi)} - 2\sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \xi_2)}{\beta - 2G(x_i, \xi)} = 0$$

where $h^{(\xi)}(.)$ denotes the derivative of h respect to ξ.

These equations cannot be solved analytically because of their nonlinear structure. The mathematical and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms. In this paper, we have used the MLE function that is under the stats4 package which could offer a numerical solution to such problems in R, R-language (26).

Let $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\xi})^T$ denotes the MLE of $\Theta = (\alpha, \beta, \xi)^T$. The distribution of $\hat{\Theta}$ under regularity conditions as $x \to \infty$ is k-variate normal distribution with mean Θ and covariance is given by the inverse of expected information matrix. This asymptotic behavior is valid if the expected information matrix is replaced by the observed information matrix. The asymptotic obtained multivariate normal distribution can be used to construct approximate confidence intervals for the individual parameters and for the hazard rate and survival functions. Package numDeriv of R language can be used to compute the Hessian matrix and its inverse, standard errors and asymptotic confidence intervals.

Conclusion

A new method to generate families of distributions has been introduced as GTL generator of distributions. Applications of four new families of distributions have been provided by using the exponential, Weibull, gamma and generalized exponential distributions as baseline distributions. The flexibility of the proposed distributions and increased range of skewness was able to fit and capture features in one real dataset much better than competitor distributions. The generation of random samples from the GTL-E, GTL-W and GTL-GE distributions is quite straightforward that is useful to perform the simulation experiments or parametric bootstrapping. The MLE method is employed to estimate the model parameters. We hope that the new family and its generated models will attract wider application in statistics.

Conflict of Interests

Authors have no conflict of interests.

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