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Original Article

Stress-Strength Reliability of Two-Parameter Exponential Distribution Based on Progressively Type-II Censored Data

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ABSTRACT

Introduction: Stress-strength models has achieved considerable attention in recent years due to its applicability in various areas like engineering, quality control, biology, genetics, medicine etc. This paper investigates estimation of the stress-strength reliability parameter R=P(Y<X) in two-parameter exponential distributions under progressively type-II censored samples.

Methods: The maximum likelihood and the best linear unbiased estimates of P(Y<X) are obtained, and the Bayes estimates of P(Y<X) are computed under the squared error, linear-exponential, and Stein loss functions. Also, confidence intervals of stress-strength reliability such as the bootstrap confidence intervals, highest posterior density credible interval, and confidence interval based on the generalized pivotal quantity are obtained.

Results: Using a simulation study, the point estimators and confidence intervals are evaluated and compared. A set of real data is presented for better clarification of the issue.

Conclusion: The results demonstrated that with increasing the sample size, in almost cases the estimated risk of all the estimators decrease. Also, in almost all cases the Bayes estimator under the linear-exponential loss function has smaller estimated risk than the other estimators. Based on our simulation, the expected lengths of all intervals tend to decrease when the sample size increases. Moreover, the highest posterior density confidence intervals are shorter than the others intervals for all the values of P(Y<X).

Introduction

In various statistical studies like lifetime tests, biological researches and clinical trials, not all sample observations are recorded, and units may not last until the experiment's completion. Progressive type-II censoring, a key method, involves removing units before failure in lifetime tests. Typically, in experiments with n independent units, only the failure times of m units are recorded, where m<n. When the first failure is observed, R_1 of the n-1 survival units is randomly removed from the experiment. In the next failure, R_2 of the n-2-R_1 units would be randomly excluded as well. Finally, when the mth failure occurs, the remaining survival units

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$$R_m = n - m - \sum_{i=1}^{m-1} R_i$$

are excluded from the experiment. In this type of censoring, m and R=(R_1,R_2,…,R_m), known as the censoring scheme, are predetermined. For further information, the readers are strongly encouraged to refer to the book authored by ([1]).

The stress-strength model examines the stability of a system against the stress applied to it, where strength and applied stress are random variables. The stress-strength reliability P(Y<X) is of great importance in reliability, and has numerous applications in genetics, nuclear physics, meteorology, economics, and medicine. For more information on the stress-strength parameter, it can be referred to ([2]).

Researchers have shown interest in stress-strength parameter estimation across various statistical distributions. Recent studies by ([3-6]), explored this estimation with different data types. Investigating stress-strength parameter estimation is crucial due to the increasing use of censored data in various scientific fields. articles,[7-9] focused on R parameter estimation in different statistical distributions with censored data.

The probability density function (pdf), and cumulative distribution function (cdf) of the two-parameter exponential distribution *E(μ,σ)* are given by

$$f(x|\mu,\sigma) = \frac{1}{\sigma} e^{\frac{-(x-\mu)}{\sigma}} \qquad x \geq \mu, \quad -\infty < \mu < \infty, \ \sigma > 0,$$

$$F(x|\mu,\sigma) = 1 - e^{\frac{-(x-\mu)}{\sigma}} \qquad x \geq \mu, \quad -\infty < \mu < \infty, \ \sigma > 0, \tag{1}$$

Let X and Y have two-parameter exponential distributions $E(\mu_1, \sigma_1)$ and $E(\mu_2, \sigma_2)$ and they are independent. Let $R=P(Y<X)$ be the stress-strength reliability. Then using (1),

$$R = P(Y < X) = \begin{cases} \dfrac{\sigma_1}{\sigma_1 + \sigma_2} e^{\frac{\mu_1 - \mu_2}{\sigma_1}} & \mu_2 \geq \mu_1 \\ 1 - \dfrac{\sigma_2}{\sigma_1 + \sigma_2} e^{\frac{\mu_2 - \mu_1}{\sigma_2}} & \mu_2 < \mu_1 \end{cases} \tag{2}$$

Many authors have inferred parameter R in exponential distributions. Point and interval estimations of stress-strength reliability have interested statisticians. In recent years, *P(Y<X)* inference in record and censored data has gained more interest. Baklizi and El-Masri[10] investigated R estimation in exponential distribution using record data. Additionally[11-13], explored stress-strength reliability estimation in two-parameter exponential distribution. Elfattah and Marwa[14] studied stress-strength parameter inference in exponential distribution with type-II censored data. Saracoglu, Kinaci[15] estimated R in one-parameter exponential distribution with progressive type-II censoring data.

Methods

In the present article, point and interval estimation of the stress-strength reliability parameter (2) is investigated under progressively type-II censored data. The parameter R is estimated with the Bayesian and ML methods. In addition, three intervals (namely bootstrap confidence, HPD credible and confidence interval based on the generalized pivotal quantity (GPQ)) are obtained.

Estimation of R

In this section, the ML, BLU and Bayes

estimates of R are obtained based on progressively Type-II censored data.

Maximum likelihood and best linear unbiased estimation

If $\boldsymbol{x}=\left(x_{1;m_1:n_1},\ldots,x_{m_1;m_1:n_1}\right)$ is a progressively Type-II censored sample from $E(\mu_1,\sigma_1)$ with scheme $R_x=\left(R_{x1},R_{x2},\ldots,R_{x,m_1}\right)$ and $\boldsymbol{y}=\left(y_{1;m_2:n_2},\ldots,y_{m_2;m_2:n_2}\right)$ is a progressively Type-II censored sample from $E(\mu_2,\sigma_2)$ with scheme $R_y=\left(R_{y1},R_{y2},\ldots,R_{ym_2}\right)$, then the likelihood function of (μ_1,σ_1) and (μ_2,σ_2) is given by ([16])

$$L(\mu_1,\mu_2,\sigma_1,\sigma_2|\boldsymbol{x},\boldsymbol{y})=$$

$$\left\{C_1\prod_{i=1}^{m_1} f\left(x_{i;m_1:n_1}\right)\left(1-F\left(x_{i;m_1:n_1}\right)\right)^{R_{ix}}\right\}\times\left\{C_2\prod_{i=1}^{m_2} f(y_{i;m_2:n_2})\left(1-F\left(y_{i;m_2:n_2}\right)\right)^{R_{iy}}\right\},$$

$$x_{m_1;m_1:n_1}>\cdots>x_{1;m_1:n_1}>\mu_1\ ,\quad y_{m_2;m_2:n_2}>\cdots>y_{1;m_2:n_2}>\mu_2 \tag{14}$$

where

$$C_1=n_1(n_1-1-R_{x1})(n_1-2-R_{x1}-R_{x2})\ldots(n_1-m_1+1-R_{x1}-\cdots-R_{xm_1-1})\ ,$$

$$C_2=n_2(n_2-1-R_{y1})(n_2-2-R_{y1}-R_{y2})\ldots(n_2-m_2+1-R_{y1}-\cdots-R_{ym_2-1}).$$

Using (1) the likelihood function (3) becomes

$$L\left(\mu_1,\mu_2,\sigma_1,\sigma_2|\boldsymbol{x},\boldsymbol{y}\right)=C_1\frac{1}{\sigma_1^{m_1}}exp\left\{-\sum_{i=1}^{m_1}\left(R_{ix}+1\right)\left(x_{i;m_1:n_1}-\mu_1\right)\Big/\sigma_1\right\}\times C_2\frac{1}{\sigma_2^{m_2}}exp\left\{-\sum_{i=1}^{m_2}\left(R_{iy}+1\right)\left(y_{i;m_2:n_2}-\mu_2\right)\Big/\sigma_2\right\}$$

Therefore, the ML estimates of the parameters $\mu_1,\mu_2,\sigma_1,\sigma_2$ can be obtained as

$$\hat{\mu}_1=x_{1;m_1:n_1}\ ,\quad \hat{\sigma}_1=\frac{1}{m_1}\sum_{i=2}^{m_1}\left(R_{ix}+1\right)\left(x_{i;m_1:n_1}-x_{1;m_1:n_1}\right),$$

$$\hat{\mu}_2=y_{1;m_2:n_2}\ ,\quad \hat{\sigma}_2=\frac{1}{m_2}\sum_{i=2}^{m_2}\left(R_{iy}+1\right)\left(y_{i;m_2:n_2}-y_{1;m_2:n_2}\right).$$

Therefore, the ML estimate of the stress-strength reliability is given by

$$\hat{R}=\begin{cases}\dfrac{\hat{\sigma}_1}{\hat{\sigma}_1+\hat{\sigma}_2}e^{\hat{\mu}_1-\hat{\mu}_2/\hat{\sigma}_1} & \hat{\mu}_2\geq\hat{\mu}_1\\ 1-\dfrac{\hat{\sigma}_2}{\hat{\sigma}_1+\hat{\sigma}_2}e^{\hat{\mu}_2-\hat{\mu}_1/\hat{\sigma}_2} & \hat{\mu}_2<\hat{\mu}_1\end{cases}$$

Also the BLU estimates of μ_1, μ_2, σ_1 and σ_2 is obtained as follows see[16]

$$\tilde{\mu}_1=x_{1;m_1:n_1}-\frac{1}{n_1}\tilde{\sigma}_1\ ,\quad \tilde{\sigma}_1=\frac{1}{m_1-1}\sum_{i=2}^{m_1}\left(R_{ix}+1\right)\left(x_{i;m_1:n_1}-x_{1;m_1:n_1}\right),$$

$$\tilde{\mu}_2=y_{1;m_1:n_1}-\frac{1}{n_2}\tilde{\sigma}_2\ ,\quad \tilde{\sigma}_2=\frac{1}{m_2-1}\sum_{i=2}^{m_2}\left(R_{iy}+1\right)\left(y_{i;m_2:n_2}-y_{1;m_2:n_2}\right).$$

Bayes estimation of R

The purpose of this section is to obtain the Bayes estimates of the parameter R. Clearly, the Bayes estimates of unknown parameters depend on the form of the loss function. The Bayes estimates are obtained based on the SE, LINEX, and Stein loss functions. The loss function $L(\theta,\delta(X))=(\delta(X)-\theta)^2$ is known as the SE loss function. Under the SE loss function, the Bayes estimator $(\delta(X))$ of the parameter θ is as follows:

$$\delta_{BS}=E(\theta\,|\,X)$$

Varian[17] proposed an asymmetric loss function known as LINEX loss function, which is defined as follows:

$$L(\theta,\delta)=e^{b(\delta-\theta)}-b(\delta-\theta)-1, \qquad b\neq 0$$

where θ is the unknown parameter, and b represents the degree of asymmetry. Under the LINEX loss function, the Bayes estimator of θ, which minimizes the posterior risk $E(L(\theta,\delta(X)) \mid X)$ is obtained as follows:

$$\delta_{BL}(\boldsymbol{X})=-\frac{1}{b}Ln\,E\left(e^{-b\theta}|\boldsymbol{X}\right),$$

The Stein loss function is a nother asymmetric loss function. This loss function is de-fined as follows:

$$L(\theta,\delta)=\frac{\delta}{\theta}-Ln\frac{\delta}{\theta}-1 .$$

The Bayes estimator under this loss function is as follows:

$$\delta_{BST}(\boldsymbol{X})=\left(E\left(\frac{1}{\theta}|\boldsymbol{X}\right)\right)^{-1},$$

If the prior distributions of the location parameters are given by

$$ð(\mu_i)=1, \qquad -\infty<\mu_i<\infty \qquad i=1,2$$

Taking the conjugate inverted gamma priors for the corresponding scale parameters

$$\pi(\sigma_i|\mu_i)=\frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)\sigma_i^{\alpha_i+1}}e^{-\beta_i/\sigma_i}$$

$$,\sigma_i>0 \quad , i=1,2,$$

then, according to equation (4) the posterior distribution for (μ_1, μ_2, σ_1, σ_2) is given by

$$\pi(\mu_1,\mu_2,\sigma_1,\sigma_2|\boldsymbol{x},\boldsymbol{y})\propto\frac{1}{\sigma_1^{\alpha_1+m_1+1}}\times$$

$$\text{üüüü}exp\left\{\frac{-1}{\sigma_1}\left(\beta_1+\sum_{i=1}^{m_1}(R_{ix}+\ )\left(x_{i;m_1:n_1}-\mu_1\right)\right)\right\}$$

$$\times\frac{1}{\sigma_2^{\alpha_2+m_2+1}}\times$$

$$\text{üüüü}exp\left\{\frac{-1}{\sigma_2}\left(\beta_2+\sum_{i=1}^{m_2}(R_{iy}+\ )\left(y_{i;m_2:n_2}-\mu_2\right)\right)\right\}$$

The conditional posterior distribution of σ_i given μ_i are given by

$$\pi_1(\mu_1|\boldsymbol{x})=\frac{n_1\left((\alpha_1+m_1-1)\left(\beta_1+\sum_{i=1}^{m_1}(R_{ix}+1)\left(x_{i;m_1:n_1}-x_{1;m_1:n_1}\right)\right)^{\alpha_1+m_1-1}\right)}{\left(\beta_1+\sum_{i=1}^{m_1}(R_{ix}+1)\left(x_{i;m_1:n_1}-\mu_1\right)\right)^{\alpha_1+m_1}}$$

$$\pi_2(\mu_2|\boldsymbol{y})=\frac{n_2\left((\alpha_2+m_2-1)\left(\beta_2+\sum_{i=1}^{m_2}(R_{iy}+1)\left(y_{i;m_2:n_2}-y_{1;m_2:n_2}\right)\right)^{\alpha_2+m_2-1}\right)}{\left(\beta_2+\sum_{i=1}^{m_2}(R_{iy}+1)\left(y_{i;m_2:n_2}-\mu_2\right)\right)^{\alpha_2+m_2}}$$

For generating of the σ_i values from inverse gamma distribution, the μ_i values are first to be generated. The μ_i values can be generated with the following two methods.

Method 1: Given the posterior density functions obtained for μ_i and used the inverse transformation algorithm, the μ_i values are generated. Let u_1 and u_2 be two independently generated uniform random values, then

$$\mu_1={}^{1}\!/_{n_1}\left(\beta_1+\sum_{i=1}^{m_1}(R_{ix}+1)x_{i;m_1:n_1}-\left(\left(\beta_1+\sum_{i=1}^{m_1}(R_{ix}+1)\left(x_{i;m_1:n_1}-x_{1;m_1:n_1}\right)\right)u_1^{-1/(\alpha_1+m_1-1)}\right)\right)$$

$$\mu_2={}^{1}\!/_{n_2}\left(\beta_2+\sum_{i=1}^{m_2}(R_{iy}+1)y_{i;m_2:n_2}-\left(\left(\beta_2+\sum_{i=1}^{m_2}(R_{iy}+1)\left(y_{i;m_2:n_2}-y_{1;m_2:n_2}\right)\right)u_2^{-1/(\alpha_2+m_2-1)}\right)\right).$$

As stated earlier, the values σ_1 and σ_2 can be generated from inverse gamma distributions using the values μ_1 and μ_2, obtained from the inverse transformation algorithm.

Method 2: The form of the posterior density functions $\pi_1(\mu_1 | x)$ and $\pi_2(\mu_2 | y)$ does not lead to an explicit Bayes estimates of the model parameters. Therefore, the values μ_1 and μ_2 can be generated from the Metropolis-Hastings method. According to the conditional posterior distribution of σ_i, the Gibbs sampling is used to generate σ_1 and σ_2.

The posterior probability density functions of μ_1 and μ_2 do not follow a known distribution, but their plots are similar to the normal probability density function plot. So, the Metropolis-Hastings method with normal proposal distribution is used to generate the values μ_1 and μ_2 from the distributions $\pi_1(\mu_1 | x)$ and $\pi_2(\mu_2 | y)$. Thus, the Gibbs sampling algorithm is described as follows:

Algorithm 1

1. Start with $\hat{\mu}_1^{(0)} = \hat{\mu}_1$ and $\hat{\mu}_2^{(0)} = \hat{\mu}_2$ as an initial guess and set *t=1*, where $\hat{\mu}_1$ and $\hat{\mu}_2$ are ML estimates of μ_1 and μ_2, respectively.
2. Using Metropolis–Hastings method, generate μ_1^t and μ_2^t from $\pi_1(\mu_1|\boldsymbol{x})$ and $\pi_2(\mu_2|\boldsymbol{y})$ with the proposal distribution as $q(\mu) \propto N(\mu^{(t-1)}, 0.1)$.
3. Generate $\sigma_1^{(t)}$ from

$$I\Gamma\left(\alpha_1 + m_1, \beta_1 + \sum_{i=1}^{m_1}(R_{ix}+1)\left(x_{i;m_1:n_1} - \mu_1^{(t)}\right)\right).$$

4. Generate $\sigma_2^{(t)}$ from

$$I\Gamma\left(\alpha_2 + m_2, \beta_2 + \sum_{i=1}^{m_2}(R_{iy}+1)\left(y_{i;m_2:n_2} - \mu_2^{(t)}\right)\right).$$

5. Compute $R^{(t)}$.
6. Set *t = t + 1*.
7. Repeat Steps 2–6, B times.

Now, the Bayes estimates of the parameter R under the SE, LINEX and Stein loss functions are as follows

$$\tilde{R}_{BS} = \hat{E}(R|\boldsymbol{x},\boldsymbol{y}) = \frac{1}{B-M}\sum_{t=M+1}^{B} R^{(t)}, \tag{5}$$

$$\tilde{R}_{BL} = -\frac{1}{b} Ln\hat{E}\left(e^{-bR}|\boldsymbol{x},\boldsymbol{y}\right) = -\frac{1}{b} Ln\left(\frac{1}{B-M}\sum_{t=M+1}^{B} e^{-bR^{(t)}}\right), \tag{6}$$

$$\tilde{R}_{BST} = \left(\hat{E}\left(\frac{1}{R}|\boldsymbol{x},\boldsymbol{y}\right)\right)^{-1} = \left(\frac{1}{B-M}\sum_{t=M+1}^{B}\frac{1}{R^{(t)}}\right)^{-1}, \tag{7}$$

respectively, where M is the burn-in period (that is, a number of iterations before the stationary distribution is achieved).

Confidence Interval for R

We used various methods to find confidence interval for the stress-strength reliability parameter R.

Bootstrap Cis

The exact distribution of the *ML* estimator of R is difficult to obtain. Therefore, it is not possible to obtain exact confidence interval of *R*. It is more appropriate to use methods such as parametric bootstrap for construction of the confidence interval *R*. This method of sampling was presented by ([18]). Before generating the parametric bootstrap samples of the parameter

R, the algorithm for generating a progressive type-II censoring sample with censoring scheme $R=(R_1,R_2,\dots,R_m)$ from two-parameter exponential distribution $E(\mu,\sigma)$ proposed by ([16]) is introduced.

Algorithm 2

Step 1. Generate m independent beta-distributed random variables $B_1, B_2, \dots, B_m$ with

$$B_j \sim Beta\left(\sum_{i=j}^{m}\left(R_i+1\right),1\right).$$

Step 2. Let $V_0=1$; calculate $V_k=B_k V_{k-1}$, $k=1, 2, \dots, m$

Step 3. Let $U_{i;m:n}=1-V_i$ for $i=1,2,\dots,m$. Then $U_{1;m:n}, U_{2;m:n},\dots,U_{m;m:n}$ is a progressively type-II censored sample of size m from $U(0,1)$ distribution.

Step 4. Finally, for given value of the parameter μ and σ, set

$$X_{i;m:n}=\mu-\sigma\ln\left(1-U_{i;m:n}\right).$$

Then $X_{1;m:n}, X_{2;m:n},\dots,X_{m;m:n}$

is the required progressively type-II censored sample from the two-parameter exponential distribution E(μ,σ).

Now, to generate parametric bootstrap samples, we can use the following algorithm.

Algorithm 3

Step 1. Compute $\hat{\mu}_1,\hat{\mu}_2,\hat{\sigma}_1,\hat{\sigma}_2$ and $\hat{R}$, the ML estimates of $\mu_1, \mu_2,\sigma_1,\sigma_2$ and R based on the original two samples of censored data x and y.

Step 2. Generate independent bootstrap censored samples $\left\{x_1^*,x_2^*,\dots,x_{m_1}^*\right\}$ and $\left\{y_1^*,y_2^*,\dots,y_{m_2}^*\right\}$ from the two-parameter exponential distribution with the parameters $\hat{\mu}_1,\hat{\sigma}_1$ and $\hat{\mu}_2,\hat{\sigma}_2$ respectively. By using these data, we compute the bootstrap estimate of *R* say $\hat{R}^*$.

Step 3. Repeat step 2, B times to obtain a set of bootstrap samples of R say $\hat{R}_1^*,\dots,\hat{R}_B^*$.

Using the above bootstrap samples of R, we can obtain three different bootstrap CIs of R as follows:

(I) Standard normal interval:

100(1-α)% bootstrap interval is the standard normal interval as

$$\left(\hat{R}-z_{\frac{\alpha}{2}}\hat{s}e_{boot},\hat{R}+z_{\frac{\alpha}{2}}\hat{s}e_{boot}\right) \qquad (8)$$

where $\hat{s}e_{boot}$ is the bootstrap estimate of the standard error based on $\hat{R}_1^*,\dots,\hat{R}_B^*$.

(II) Percentile bootstrap (Boot-p) interval[19]:

Let $G(x)=P\left(\hat{R}^*\le x\right)$ be the cdf of $\hat{R}^*$. Define $\hat{R}_{Bp}=G^{-1}(x)$ for a given *x*. Then approximate 100(1-α)% confidence interval for R is given by

$$\left(\hat{R}_{Bp}\left(\frac{\alpha}{2}\right),\hat{R}_{Bp}\left(1-\frac{\alpha}{2}\right)\right), \qquad (9)$$

that is, just use the $\left(\frac{\alpha}{2}\right)$ and $\left(1-\frac{\alpha}{2}\right)$ quantiles of the bootstrap sample $\hat{R}_1^*,\dots,\hat{R}_B^*$.

(III) Student's t bootstrap (Boot-t) interval [20]:

Let

$$T_b^*=\frac{\left(\hat{R}_b^*-\hat{R}\right)}{\hat{s}e_b^*}, \quad b=1,\dots,B$$

where $\hat{s}e_b^*$ is an estimate of the standard error of $\hat{R}_b^*$. Then 100(1-α)% bootstrap student's t interval is given by

$$(\hat{R}-t^{*}_{1-\frac{\alpha}{2}}\hat{s}e_{boot},\hat{R}-t^{*}_{\frac{\alpha}{2}}\hat{s}e_{boot}), \quad (10)$$

where t^{*}_{α} is the α quantile of $\hat{T}^{*}_{1},\ldots,\hat{T}^{*}_{B}$..

HPD Credible Interval

In this section, we use the Bayesian analysis to obtain HPD credible interval of R. Due to the fact that the construction of HPD credible interval is usually difficult and the endpoint often lacks simple closed forms, numerical methods, including possibly numerical integration and numerical root finding, are used. Chen and Shao[21] presented a method for estimating the endpoints of the Bayesian intervals that easy to implement and has asymptotically the desired probability content. Let $R^{(t)}, t=M+1,\ldots,B$ be the posterior sample generated by Algorithm 1. Then from[21], the 100(1-α) % HPD credible interval is given by the shortest interval of the form $(R_{(i)},R_{(i+[(1-\alpha)B])})$, where $R_{[(1-\alpha)B]}$ is the [(1-α) B]th smallest integer of $\{R^{(t)},t=M+1,\ldots,B\}$.

Confidence interval based on GPQ

The confidence interval based on the GPQ was first introduced by Tsui and Weerahandi,[22] and subsequent research by Weerhandi[23] expanded on this topic. The GPQ can be useful when the pivotal quantity is complex or its distribution is unknown. More information on the GPQ can be found in Weerhandi's book.[24]

In this section, the generalized pivotal quantities for the parameters μ_1,σ_1 and μ_2,σ_2 are obtained, using which the confidence interval for the stress-strength parameter is then constructed.

Generalized pivotal quantity

Let $x = (x_1,x_2,\ldots,x_n)$ be the observation of a random vector $X=(X_1, X_2,\ldots,X_n)$ which has a distribution depends on the parameters μ, σ. To construct a generalized confidence interval (GCI) for μ,σ, first define a GPQ, $Q(X;x,\mu,\sigma)$, which is a function of the random vector X, its observed value x and the parameters μ,σ. Generally, $Q(X;x,\mu,\sigma)$ is a GPQ for μ,σ, if it satisfy the following two conditions:

1. For a given x, the distribution of $Q(X;x,\mu,\sigma)$ is free from unknown parameters.
2. The value of $Q(X;x,\mu,\sigma)$ at x should be μ,σ.

Now if $Q_{\beta}(X;x,\mu,\sigma)$ is a βth percentile of $Q(X;x,\mu,\sigma)$ distribution, then the 100(1-α)% GCI of μ,σ is any value of μ,σ that satisfies

$$\begin{pmatrix} Q_{\frac{\alpha}{2}}\left(\boldsymbol{X};\boldsymbol{x},\mu,\sigma\right)\le Q\left(\boldsymbol{X};\boldsymbol{x},\mu,\sigma\right)\le Q_{1-\frac{\alpha}{2}} \\ \left(\boldsymbol{X};\boldsymbol{x},\mu,\sigma\right) \end{pmatrix}=1-\alpha \quad (11)$$

To find a $100(1-\alpha)\%$ GCI of R, the GPQ for μ_1,σ_1 and μ_2,σ_2 in two-parameter exponential distribution should be found.

The ML estimators of the parameters of the two-parameter exponential distribution based on progressively Type-II censored data are:

$$\hat{\mu}_1 = X_{1;m_1:n_1}\ ,\ \ \hat{\sigma}_1 = \frac{1}{m_1}\sum_{i=2}^{m_1}\left(R_{ix}+1\right)\left(X_{i;m_1:n_1}-X_{1;m_1:n_1}\right),$$

$$\hat{\mu}_2 = Y_{1;m_2:n_2}\ ,\ \ \hat{\sigma}_2 = \frac{1}{m_2}\sum_{i=2}^{m_2}\left(R_{iy}+1\right)\left(Y_{i;m_2:n_2}-Y_{1;m_2:n_2}\right)$$

It can be shown that

$$\frac{2n_1(\hat{\mu}_1-\mu_1)}{\sigma_1}\sim\chi^2(2),\quad \frac{2m_1\hat{\sigma}_1}{\sigma_1}\sim\chi^2(2m_1-2)$$
$$\frac{2n_2(\hat{\mu}_2-\mu_2)}{\sigma_2}\sim\chi^2(2),\quad \frac{2m_2\hat{\sigma}_2}{\sigma_2}\sim\chi^2(2m_2-2)$$

Suppose $X=X_{1;m_1:n_1},\dots,X_{m_1;m_1:n_1}$ be a progressively type-II censored sample from $E(\mu 1,\sigma_1)$ distribution. Let $\hat{\mu}_{1,0}$ and $\hat{\sigma}_{1,0}$ be a fixed observed value of $\hat{\mu}_1$ and $\hat{\sigma}_1$, then

$$Q_{\mu_1}=\hat{\mu}_{1,0}-\frac{2n_1(\hat{\mu}_1-\mu_1)}{2n_1\sigma_1}\times\frac{2m_1\sigma_1}{2m_1\hat{\sigma}_1}\times\hat{\sigma}_{1,0} \qquad (12)$$

$$=\hat{\mu}_{1,0}-\frac{m_1}{n_1}\times\frac{\chi^2(2)}{\chi^2(2m_1-2)}\times\hat{\sigma}_{1,0}$$

$$Q_{\sigma_1}=\frac{\sigma_1}{2m_1\hat{\sigma}_1}\times 2m_1\hat{\sigma}_{1,0}=\frac{2m_1\hat{\sigma}_{1,0}}{\chi^2(2m_1-2)},$$

are the GPQ for μ_1 and σ_1, respectively, since
1. The distribution of Q_{μ_1} and Q_{σ_1}, does not depend on any unknown parameters.
2. When $(\hat{\mu}_1,\hat{\sigma}_1)=(\hat{\mu}_{1,0},\hat{\sigma}_{1,0})$ then $Q_{\mu_1}=\mu_1$ and $Q_{\sigma_1}=\sigma_1$.
Also, suppose $Y=(Y_{1;m_2:n_2},\dots,Y_{m_2;m_2:n_2}$ is a progressively Type-II censored sample from $E(\mu_2,\sigma_2)$ distribution. Similarly, the GPQ for μ_2 and σ_2 are given by

$$Q_{\mu_2}=\hat{\mu}_{2,0}-\frac{2n_2(\hat{\mu}_2-\mu_2)}{2n_2\sigma_2}\times\frac{2m_2\sigma_2}{2m_2\hat{\sigma}_2}\times\hat{\sigma}_{2,0} \qquad (13)$$

$$=\hat{\mu}_{2,0}-\frac{m_2}{n_2}\times\frac{\chi^2(2)}{\chi^2(2m_2-2)}\times\hat{\sigma}_{2,0},$$

$$Q_{\sigma_2}=\frac{\sigma_2}{2m_2\hat{\sigma}_2}\times 2m_2\hat{\sigma}_{2,0}=\frac{2m_2\hat{\sigma}_{2,0}}{\chi^2(2m_2-2)},$$

respectively, where $\hat{\mu}_2,\hat{\sigma}_2$ is the ML estimators of μ_2,σ_2 and $(\hat{\mu}_{2,0},\hat{\sigma}_{2,0})$ are ML estimates of μ_2,σ_2 based on the observed sample $y=(y_{1;m_2:n_2},\dots,y_{m_2;m_2:n_2})$.
Based on the above results, the GPQ for R can be found by replacing the unknown parameters in (2) by (12) and (13). Therefore

$$Q_R=\begin{cases}\dfrac{Q_{\sigma_1}}{Q_{\sigma_1}+Q_{\sigma_2}}e^{\frac{Q_{\mu_1}-Q_{\mu_2}}{Q_{\sigma_1}}} & Q_{\mu_2}\ge Q_{\mu_1}\\[2ex] 1-\dfrac{Q_{\sigma_2}}{Q_{\sigma_1}+Q_{\sigma_2}}e^{\frac{Q_{\mu_2}-Q_{\mu_1}}{Q_{\sigma_2}}} & Q_{\mu_2}<Q_{\mu_1}\end{cases}$$

It is clear that the distribution of Q_R is very complicated. Therefore, it is not possible to obtain exact confidence interval of *R*, but for each $(\hat{\mu}_{(1,0)},\hat{\sigma}_{(1,0)},\hat{\mu}_{(2,0)},\hat{\sigma}_{(2,0)})$ the distribution of Q_R, does not depend on any unknown parameters. As a result, the GCI of *R* can be found by a Monte Carlo simulation through the following algorithm:

Algorithm 4

Step 1. For a given censored data, compute the ML estimates μ_1, μ_2, σ_1 and σ_2 (*i.e.*, $\hat{\mu}_{1,0},\hat{\mu}_{2,0},\hat{\sigma}_{(1,0)}$ and $\hat{\sigma}_{(2,0)}$).

Step 2. Generate data from the distributions $\chi^2(2)$, $\chi^2(2m_1-2)$ and $\chi^2(2m_2-2)$.

Step 3. Compute Q_{μ_1}, Q_{σ_1}, Q_{μ_2}, Q_{σ_2} and Q_R.

Step 4. Repeat steps 2 and 3, B times and order values of Q_R.

Step 5. The lower and upper confidence limits for R are given by $Q_R^{\left[\left(\frac{\alpha}{2}\right)B\right]}$ and $Q_R^{\left[\left(1-\frac{\alpha}{2}\right)B\right]}$, respectively,
where $Q_R^{[(\alpha)B]}$ is the α quantile of $Q_R^{[1]},\dots,Q_R^{[B]}$.

Results

Simulation study

The point estimators and CIs proposed in the paper are compared using a Monte Carlo simulation study. Average, absolute biases and the estimated risk (ER) of the ML, BLU, GPQ and the approximate Bayes estimators of R are reported based on 10000 iterations. Furthermore, coverage probability and the expected lengths of the CIs are presented and compared to each other in this simulation. The simulated absolute bias and risk of the estimator of *R* under SE loss function are obtained as

$$\hat{R}_i = \frac{1}{10000}\sum_{i=1}^{10000} R^{(i)},\ Absolute\,Bias = \frac{1}{10000}\sum_{i=1}^{10000}\left|\hat{R}_i - R\right|,$$
$$ER_{BS}(R) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{R}_i - R\right)^2,$$

where R_i is the estimate of *R* in ith replication of simulation. In the case of Bayes estimation, $\hat{R}_i$ is an approximate Bayes estimator of *R* in *i*th replication. Moreover, the ER of *R* under the LINEX and Stein loss functions are given by

$$ER_{BL}(R) = \frac{1}{10000}\sum_{i=1}^{10000}\left(e^{b\left(\hat{R}_i - R\right)} - b\left(\hat{R}_i - R\right) - 1\right),$$

$$ER_{BST}(R) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\frac{\hat{R}_i}{R} - Ln\frac{\hat{R}_i}{R} - 1\right),$$

respectively. To compute the Bayes estimates and HPD credible intervals, the two prior density functions are considered as follows:

Prior 1: $\alpha_1=\alpha_2=2$, $\beta_1=\beta_2=3$
Prior 2: $\alpha_1=\alpha_2=3$, $\beta_1=\beta_2=5$

Three sets of parameter values ($\mu_1=1,\mu_2=2,\sigma_1=\sigma_2=2$), ($\mu_1=2,\mu_2=2,\sigma_1=\sigma_2=2$) and ($\mu_1=3,\mu_2=2,\sigma_1=\sigma_2=2$), are used for comparison of different methods and censoring schemes. Therefore, from equation (2), R=0.3032,0.5,0.6967. Also, 10000 datasets are generated for each set of parameter values, which is utilized for comparison between the Bayes, ML and BLU estimates. Moreover, different confidence intervals, such as bootstrap CIs, GCIs and the HPD credible intervals based on inverse transformation algorithm and Gibbs sampling technique are obtained. CIs are compared in terms of coverage probabilities and expected lengths.

The three censoring schemes given in Tables 1-3 are used in the simulation study. For convenience, the notation 0^{*k} for *k* successive zeroes is introduced.

Table 1. First censoring schemes

	(*m*,*n*)	C. S.
r_1	(5,15)	(0,0,0,0,10)
r_2	(5,15)	(2,2,2,2,2)
r_3	(5,15)	(10,0,0,0,0)

Table 2. Second censoring schemes

	(*m*,*n*)	C. S.
r_1	(10,30)	(0,0,0,0,0,0,0,0,0,20)
r_2	(10,30)	(2,2,2,2,2,2,2,2,2,2)
r_3	(10,30)	(20,0,0,0,0,0,0,0,0,0)

Table 3. Third censoring schemes

	(*m*,*n*)	C. S.
r_1	(15,30)	(0^{*14},30)
r_2	(15,30)	(2^{*15})
r_3	(15,30)	(30,0^{*14})

The ML, BLU, and GPQ estimates are firstly obtained for the parameter R and their average estimates, absolute biases, and then ERs are calculated. Next, the Bayesian estimates are obtained based on the SE, LINEX with b=1, and Stein loss function using the Metropolis-Hastings within Gibbs sampling method, and the absolute biases and ERs of the approximate Bayes estimators based on Priors 1 and 2 are calculated. The coverage probability (CP) and expected length (EL) of 95% CIs for R such as bootstrap CIs namely the percentile interval (Boot-p) based on ML and BLU estimates, GCIs and HPD credible intervals are calculated. The HPD credible intervals under the SE loss function are constructed based on inverse transformation and Gibbs sampling, and the HPD interval for the LINEX and Stein loss functions are obtained using the Gibbs sampling method. The CIs of bootstrap and GPQ are obtained from 10000 iterations. The Bayesian estimate and HPD interval are calculated based on 10000 samples and discard the first 2000 values as burn-in period. The results are given in Tables 4 and 5.

Note that the censoring schemes $(r_i,r_j),i,j=1,2,3$ are determined by Tables 1-3 according to (m_1,m_2) sample sizes of observed x and y.

From the simulation study, the following results obtained from Table 4:

1- The ERs of all estimators decreases when the sample size (m_1,m_2) increases. But, when the censoring schemes of observations x and y are different (i.e., $(r_i,r_j),i\neq j$), in almost all cases the ERs of the Bayes estimators increases as the sample size increases.

2- Under each fixed censoring scheme and sample size, in almost all cases the ERs of the Bayes estimators under the SE and LINEX loss functions are increasing function of R when the censoring schemes of observations x and y are different. Also the ERs of the Bayes estimators under the Stein loss function are decreasing function of R. Moreover, in this case the ERs of the ML and BLU estimators of R decreases when the value of R getting away from 0.5.

3- In almost all cases, the Bayes estimator under the LINEX loss function has smaller ER than the other estimators and the absolute bias of BLU estimator is less than the ML and Bayes estimators under SE loss function.

4- The Bayes estimators are sensitive to the values of the parameters of prior distributions.

Table 4. Average estimates (AVR), absolute biases and ERs of the estimators of

(m_1,m_2)	R	C.S	ML			BLU			Bayes - LINEX (Prior 1)		Bayes - LINEX (Prior 2)	
			AVR	A .Bias	ER	AVR	A .Bias	ER	AVR	ER	AVR	ER
(5,5)	0.3032	(r_1,r_1)	0.2237	0.0795	0.0249	0.3225	0.0193	0.0191	0.3033	0.0053	0.3080	0.0039
		(r_2,r_2)	0.2325	0.0707	0.0239	0.3141	0.0108	0.0197	0.3004	0.0047	0.3013	0.0038
		(r_3,r_3)	0.2251	0.0782	0.0243	0.2981	0.0051	0.0168	0.3099	0.0049	0.2995	0.0034
		(r_1,r_2)	0.2221	0.0812	0.0251	0.3177	0.0144	0.0187	0.3145	0.0063	0.3151	0.0046
		(r_1,r_3)	0.2230	0.0802	0.0245	0.3303	0.0271	0.0200	0.3199	0.0095	0.3216	0.0067
		(r_2,r_3)	0.2297	0.0735	0.0238	0.3259	0.0227	0.0187	0.3159	0.0084	0.3113	0.0059
	0.5	(r_1,r_1)	0.4933	0.0067	0.0374	0.5016	0.0016	0.0203	0.4920	0.0056	0.4972	0.0044
		(r_2,r_2)	0.4942	0.0058	0.0366	0.5011	0.0011	0.0193	0.4948	0.0056	0.4915	0.0043
		(r_3,r_3)	0.4955	0.0045	0.0341	0.4901	0.0099	0.0185	0.4952	0.0053	0.4913	0.0038
		(r_1,r_2)	0.5033	0.0033	0.0329	0.4981	0.0019	0.0195	0.5041	0.0069	0.5091	0.0054
		(r_1,r_3)	0.4903	0.0097	0.0345	0.5354	0.0354	0.0196	0.5098	0.0160	0.5119	0.0107
		(r_2,r_3)	0.4968	0.0032	0.0321	0.5186	0.0186	0.0185	0.5124	0.0114	0.5109	0.0078
	0.6967	(r_1,r_1)	0.7652	0.0685	0.0255	0.6809	0.0157	0.0193	0.6852	0.0049	0.6856	0.0041
		(r_2,r_2)	0.7749	0.0782	0.0241	0.6859	0.0109	0.0195	0.6864	0.0051	0.6845	0.0038
		(r_3,r_3)	0.7730	0.0762	0.0241	0.7045	0.0078	0.0158	0.6865	0.0048	0.6843	0.0035
		(r_1,r_2)	0.7662	0.0698	0.0243	0.6859	0.0108	0.0178	0.7154	0.0064	0.7198	0.0045
		(r_1,r_3)	0.7680	0.0713	0.0232	0.7151	0.0183	0.0177	0.7179	0.0141	0.7208	0.0117
		(r_2,r_3)	0.7668	0.0701	0.0233	0.7085	0.0118	0.0165	0.7112	0.0112	0.7176	0.0075

Table 4. (Continued)

(m_1,m_2)	R	C.S	ML			BLU			Bayes - LINEX (Prior 1)		Bayes - LINEX (Prior 2)	
			AVR	A .Bias	ER	AVR	A .Bias	ER	AVR	ER	AVR	ER
(10,10)	0.3032	(r_1, r_1)	0.2649	0.0383	0.0122	0.3107	0.0074	0.0106	0.3027	0.0035	0.2991	0.0031
		(r_2, r_2)	0.2678	0.0354	0.0120	0.3106	0.0073	0.0107	0.3055	0.0037	0.3055	0.0032
		(r_3, r_3)	0.2625	0.0407	0.0125	0.3057	0.0025	0.0102	0.2991	0.0038	0.3035	0.0031
		(r_1, r_2)	0.2634	0.0398	0.0113	0.3168	0.0135	0.0101	0.3043	0.0058	0.3075	0.0046
		(r_1, r_3)	0.2689	0.0343	0.0122	0.3206	0.0173	0.0104	0.3088	0.0132	0.3067	0.0111
		(r_2, r_3)	0.2661	0.0371	0.0118	0.3159	0.0127	0.0108	0.3124	0.0097	0.3096	0.0078
	0.5	(r_1, r_1)	0.4993	0.0007	0.0148	0.4991	0.0009	0.0117	0.4970	0.0040	0.4925	0.0034
		(r_2, r_2)	0.5027	0.0027	0.0144	0.5006	0.0006	0.0116	0.4986	0.0038	0.4985	0.0035
		(r_3, r_3)	0.4989	0.0011	0.0147	0.5041	0.0041	0.0107	0.4925	0.0042	0.4988	0.0034
		(r_1, r_2)	0.5005	0.0005	0.0134	0.5063	0.0063	0.0110	0.5071	0.0086	0.5058	0.0067
		(r_1, r_3)	0.5031	0.0031	0.0145	0.5154	0.0154	0.0109	0.5098	0.0297	0.5103	0.0236
		(r_2, r_3)	0.5033	0.0033	0.0146	0.5130	0.0130	0.0112	0.5067	0.0190	0.5093	0.0146
	0.6967	(r_1, r_1)	0.7347	0.0379	0.0124	0.6872	0.0094	0.0109	0.6918	0.0038	0.6872	0.0032
		(r_2, r_2)	0.7365	0.0398	0.0123	0.6899	0.0067	0.0109	0.6915	0.0035	0.6921	0.0032
		(r_3, r_3)	0.7355	0.0387	0.0123	0.7024	0.0057	0.0095	0.6879	0.0039	0.6902	0.0033
		(r_1, r_2)	0.7359	0.0392	0.0114	0.6939	0.0027	0.0102	0.7151	0.0089	0.7147	0.0074
		(r_1, r_3)	0.7372	0.0405	0.0119	0.7072	0.0105	0.0099	0.7097	0.0311	0.7107	0.0252
		(r_2, r_3)	0.7393	0.0426	0.0127	0.7072	0.0105	0.0099	0.7149	0.0208	0.7164	0.0163
(15,15)	0.3032	(r_1, r_1)	0.2814	0.0218	0.0079	0.3132	0.0099	0.0071	0.3006	0.0029	0.3046	0.0026
		(r_2, r_2)	0.2818	0.0214	0.0079	0.3074	0.0041	0.0067	0.2979	0.0029	0.3026	0.0026
		(r_3, r_3)	0.2759	0.0273	0.0081	0.3069	0.0037	0.0070	0.3051	0.0028	0.3010	0.0025
		(r_1, r_2)	0.2793	0.0240	0.0082	0.3097	0.0064	0.0074	0.3086	0.0056	0.3076	0.0047
		(r_1, r_3)	0.2787	0.0245	0.0077	0.3153	0.0121	0.0078	0.3101	0.0157	0.3089	0.0133
		(r_2, r_3)	0.2773	0.0259	0.0083	0.3134	0.0102	0.0075	0.3052	0.0099	0.3077	0.0083
	0.5	(r_1, r_1)	0.5061	0.0061	0.0095	0.5054	0.0054	0.0075	0.4956	0.0032	0.4997	0.0028
		(r_2, r_2)	0.5059	0.0059	0.0092	0.4994	0.0056	0.0071	0.4921	0.0031	0.4957	0.0027
		(r_3, r_3)	0.4973	0.0027	0.0091	0.5052	0.0052	0.0074	0.4948	0.0032	0.4954	0.0028
		(r_1, r_2)	0.5037	0.0037	0.0094	0.5019	0.0019	0.0081	0.5071	0.0093	0.5078	0.0073
		(r_1, r_3)	0.5018	0.0018	0.0089	0.5105	0.0105	0.0083	0.5165	0.0393	0.5197	0.0321
		(r_2, r_3)	0.5004	0.0004	0.0094	0.5114	0.0114	0.0080	0.5123	0.0222	0.5146	0.0186
	0.6967	(r_1, r_1)	0.7279	0.0312	0.0087	0.6972	0.0004	0.0069	0.6920	0.0030	0.6946	0.0026
		(r_2, r_2)	0.7274	0.0306	0.0082	0.6920	0.0047	0.0065	0.6880	0.0029	0.6895	0.0025
		(r_3, r_3)	0.7196	0.0228	0.0078	0.7029	0.0062	0.0065	0.6937	0.0028	0.6924	0.0024
		(r_1, r_2)	0.7269	0.0302	0.0085	0.6934	0.0033	0.0075	0.7111	0.0098	0.7018	0.0085
		(r_1, r_3)	0.7241	0.0273	0.0079	0.7048	0.0081	0.0073	0.7213	0.0399	0.7189	0.0324
		(r_2, r_3)	0.7237	0.0270	0.0083	0.7074	0.0106	0.0072	0.7254	0.0238	0.7217	0.0208
(5,5)	0.3032	(r_1, r_1)	0.3080	0.0047	0.0101	0.3046	0.0014	0.0078	0.2523	0.1252	0.2622	0.0712
		(r_2, r_2)	0.3068	0.0036	0.0099	0.3081	0.0048	0.0071	0.2549	0.1156	0.2636	0.0726
		(r_3, r_3)	0.3082	0.0049	0.0097	0.3077	0.0045	0.0079	0.2537	0.1243	0.2635	0.0775
		(r_1, r_2)	0.3211	0.0179	0.0133	0.3234	0.0202	0.0093	0.2856	0.1017	0.2959	0.0584
		(r_1, r_3)	0.3269	0.0237	0.0201	0.3215	0.0183	0.0141	0.3425	0.0793	0.3294	0.0537
		(r_2, r_3)	0.3157	0.0125	0.0151	0.3198	0.0166	0.0117	0.3177	0.0830	0.3136	0.0509
	0.5	(r_1, r_1)	0.4991	0.0008	0.0103	0.4965	0.0035	0.0087	0.4653	0.0341	0.4692	0.0224
		(r_2, r_2)	0.4977	0.0022	0.0108	0.5017	0.0017	0.0081	0.4659	0.0319	0.4702	0.0235
		(r_3, r_3)	0.4988	0.0012	0.0115	0.4972	0.0028	0.0092	0.4635	0.0335	0.4682	0.0249
		(r_1, r_2)	0.5086	0.0086	0.0144	0.5095	0.0095	0.0108	0.5348	0.0244	0.5291	0.0182
		(r_1, r_3)	0.5199	0.0199	0.0314	0.5168	0.0168	0.0222	0.5347	0.0408	0.5311	0.0279
		(r_2, r_3)	0.5077	0.0077	0.0181	0.5087	0.0087	0.0146	0.5280	0.0293	0.5288	0.0221
	0.6967	(r_1, r_1)	0.6909	0.0057	0.0094	0.6889	0.0077	0.0077	0.6689	0.0143	0.6688	0.0099
		(r_2, r_2)	0.6893	0.0073	0.0097	0.6932	0.0035	0.0071	0.6694	0.0134	0.6695	0.0105
		(r_3, r_3)	0.6901	0.0065	0.0103	0.6878	0.0088	0.0083	0.6665	0.0139	0.6666	0.0109
		(r_1, r_2)	0.7189	0.0222	0.0123	0.7145	0.0178	0.0092	0.7265	0.0109	0.7256	0.0074
		(r_1, r_3)	0.7311	0.0344	0.0333	0.7257	0.0190	0.0232	0.7327	0.0265	0.7301	0.0176
		(r_2, r_3)	0.7278	0.0311	0.0242	0.7149	0.0182	0.0156	0.7290	0.0175	0.7279	0.0119
(10,10)	0.3032	(r_1, r_1)	0.3066	0.0034	0.0074	0.3062	0.0029	0.0064	0.2785	0.0581	0.2775	0.0503
		(r_2, r_2)	0.3068	0.0036	0.0078	0.3133	0.0101	0.0063	0.2732	0.0629	0.2751	0.0510
		(r_3, r_3)	0.3097	0.0065	0.0076	0.3054	0.0021	0.0069	0.2781	0.0629	0.2778	0.0501
		(r_1, r_2)	0.3101	0.0069	0.0108	0.3089	0.0057	0.0095	0.3253	0.0531	0.3220	0.0433
		(r_1, r_3)	0.3179	0.0147	0.0272	0.3168	0.0136	0.0218	0.3298	0.0892	0.3285	0.0682
		(r_2, r_3)	0.3127	0.0095	0.0184	0.3156	0.0124	0.0141	0.3289	0.0621	0.3257	0.0506

Table 4. (Continued)

(m_1,m_2)	R	C.S	ML			BLU			Bayes - LINEX (Prior 1)		Bayes - LINEX (Prior 2)	
			AVR	A .Bias	ER	AVR	A .Bias	ER	AVR	ER	AVR	ER
(10,10)	0.5	(r_1, r_1)	0.5022	0.0022	0.0078	0.5005	0.0005	0.0068	0.4761	0.0205	0.4823	0.0172
		(r_2, r_2)	0.4999	0.0001	0.0082	0.5076	0.0076	0.0067	0.4785	0.0201	0.4786	0.0171
		(r_3, r_3)	0.5041	0.0041	0.0083	0.5004	0.0004	0.0074	0.4843	0.0202	0.4803	0.0172
		(r_1, r_2)	0.5167	0.0167	0.0156	0.5187	0.0187	0.0136	0.5238	0.0252	0.5211	0.0195
		(r_1, r_3)	0.5243	0.0234	0.0569	0.5214	0.0214	0.0442	0.5314	0.0824	0.5298	0.0618
		(r_2, r_3)	0.5187	0.0187	0.0353	0.5135	0.0135	0.0266	0.5289	0.0482	0.5278	0.0398
	0.6967	(r_1, r_1)	0.6984	0.0017	0.0074	0.6951	0.0016	0.0063	0.6757	0.0094	0.6828	0.0081
		(r_2, r_2)	0.6843	0.0024	0.0073	0.7002	0.0035	0.0062	0.6788	0.0092	0.6778	0.0079
		(r_3, r_3)	0.6936	0.0031	0.0078	0.6929	0.0038	0.0057	0.6814	0.0096	0.6779	0.0075
		(r_1, r_2)	0.7124	0.0157	0.0165	0.7123	0.0156	0.0142	0.7142	0.0144	0.7108	0.0115
		(r_1, r_3)	0.7201	0.0234	0.0579	0.7189	0.0222	0.0478	0.7282	0.0481	0.7246	0.0395
		(r_2, r_3)	0.7173	0.0206	0.0374	0.7146	0.0179	0.0310	0.7160	0.0311	0.7157	0.0257
(15,15)	0.3032	(r_1, r_1)	0.3077	0.0045	0.0060	0.3071	0.0039	0.0051	0.2852	0.0397	0.2851	0.0358
		(r_2, r_2)	0.3030	0.0002	0.0059	0.3067	0.0035	0.0048	0.2837	0.0412	0.2813	0.0377
		(r_3, r_3)	0.2990	0.0042	0.0058	0.3051	0.0018	0.0046	0.2774	0.0467	0.2803	0.0393
		(r_1, r_2)	0.3102	0.0070	0.0102	0.3105	0.0073	0.0093	0.3169	0.0431	0.3179	0.0375
		(r_1, r_3)	0.3179	0.0147	0.0291	0.3189	0.0157	0.0248	0.3287	0.0969	0.3296	0.0825
		(r_2, r_3)	0.3126	0.0094	0.0198	0.3137	0.0105	0.0161	0.3232	0.0652	0.3221	0.0569
	0.5	(r_1, r_1)	0.5050	0.0050	0.0065	0.5030	0.0030	0.0054	0.4878	0.0135	0.4874	0.0127
		(r_2, r_2)	0.4984	0.0016	0.0063	0.5007	0.0007	0.0051	0.4881	0.0138	0.4843	0.0137
		(r_3, r_3)	0.4953	0.0047	0.0064	0.4967	0.0033	0.0053	0.4793	0.0157	0.4900	0.0121
		(r_1, r_2)	0.5059	0.0059	0.0166	0.5069	0.0069	0.0152	0.5117	0.0403	0.5145	0.0222
		(r_1, r_3)	0.5231	0.0231	0.0611	0.5219	0.0219	0.0573	0.5295	0.0815	0.5289	0.0819
		(r_2, r_3)	0.5125	0.0125	0.0405	0.5112	0.0112	0.0363	0.5243	0.0592	0.5208	0.0514
	0.6967	(r_1, r_1)	0.7011	0.0044	0.0060	0.6984	0.0017	0.0049	0.6863	0.0066	0.6864	0.0059
		(r_2, r_2)	0.6944	0.0022	0.0057	0.6948	0.0019	0.0047	0.6883	0.0064	0.6844	0.0065
		(r_3, r_3)	0.6957	0.0010	0.0056	0.6936	0.0031	0.0050	0.6884	0.0063	0.6811	0.0062
		(r_1, r_2)	0.7197	0.0230	0.0196	0.7167	0.0200	0.0172	0.7182	0.0165	0.7168	0.0147
		(r_1, r_3)	0.7389	0.0422	0.0674	0.7374	0.0407	0.0594	0.7378	0.0559	0.7308	0.0493
		(r_2, r_3)	0.7218	0.0251	0.0450	0.7268	0.0301	0.0387	0.7298	0.0377	0.7268	0.0331

Table 5. Expected lengths (EL) and coverage probability (CP) of the confidence intervals with

(m_1,m_2)	R	C.S	Boot-p (ML)		Boot-p (BLU)		Bayes Prior1				Bayes Prior2			
							I.T - SE		MCMC - SE		I.T - SE		MCMC - SE	
			EL	CP	EL	CP	EL	CP	EL	CP	EL	CP	EL	CP
(5,5)	0.3032	(r_1, r_1)	0.4645	0.822	0.4568	0.911	0.4459	0.962	0.4485	0.953	0.4278	0.981	0.4244	0.978
		(r_2, r_2)	0.4749	0.840	0.4630	0.900	0.4470	0.960	0.4489	0.958	0.4245	0.981	0.4279	0.985
		(r_3, r_3)	0.4674	0.834	0.5142	0.933	0.4464	0.965	0.4501	0.962	0.4251	0.979	0.4271	0.981
		(r_1, r_2)	0.4623	0.803	0.4589	0.905	0.4700	0.968	0.4678	0.951	0.4469	0.981	0.4468	0.978
		(r_1, r_3)	0.4662	0.818	0.4773	0.906	0.4841	0.936	0.4844	0.930	0.4584	0.954	0.4598	0.951
		(r_2, r_3)	0.4741	0.850	0.4858	0.921	0.4790	0.947	0.4813	0.946	0.4522	0.962	0.4539	0.961
	0.5	(r_1, r_1)	0.5931	0.869	0.4969	0.921	0.4737	0.972	0.4733	0.963	0.4489	0.974	0.4487	0.972
		(r_2, r_2)	0.5962	0.888	0.5457	0.935	0.4720	0.958	0.4726	0.970	0.4489	0.977	0.4505	0.984
		(r_3, r_3)	0.6002	0.895	0.5853	0.982	0.4720	0.966	0.4717	0.952	0.4495	0.980	0.4490	0.973
		(r_1, r_2)	0.6049	0.900	0.5060	0.932	0.4667	0.941	0.4659	0.939	0.4447	0.960	0.4457	0.961
		(r_1, r_3)	0.6005	0.897	0.5397	0.940	0.4338	0.923	0.4330	0.938	0.4263	0.939	0.4248	0.944
		(r_2, r_3)	0.6063	0.907	0.5489	0.959	0.4498	0.933	0.4826	0.937	0.4363	0.929	0.4390	0.932
	0.6967	(r_1, r_1)	0.4751	0.831	0.4554	0.916	0.4457	0.968	0.4505	0.963	0.4235	0.981	0.4278	0.978
		(r_2, r_2)	0.4706	0.837	0.4653	0.915	0.4507	0.968	0.4505	0.978	0.4250	0.979	0.4285	0.990
		(r_3, r_3)	0.4725	0.845	0.5154	0.946	0.4472	0.953	0.4492	0.965	0.4249	0.983	0.4276	0.978
		(r_1, r_2)	0.4774	0.832	0.4649	0.921	0.3394	0.928	0.4001	0.911	0.3859	0.948	0.3886	0.949
		(r_1, r_3)	0.4779	0.844	0.4894	0.920	0.2858	0.912	0.2922	0.909	0.3136	0.922	0.3159	0.917
		(r_2, r_3)	0.4798	0.859	0.4982	0. 933	0.3380	0.925	0.3324	0.915	0.3476	0.921	0.3510	0.919
(10,10)	0.3032	(r_1, r_1)	0.3824	0.887	0.3653	0.921	0.3549	0.936	0.3573	0.950	0.3443	0.948	0.3445	0.964
		(r_2, r_2)	0.3844	0.901	0.3685	0.924	0.3542	0.945	0.3568	0.943	0.3422	0.951	0.3476	0.975
		(r_3, r_3)	0.3802	0.889	0.3909	0.949	0.3501	0.929	0.3576	0.948	0.3424	0.956	0.3434	0.946

Table 5. (continued)

(m_1,m_2)	R	C.S	Boot-p (ML)		Boot-p (BLU)		Bayes				Bayes			
							Prior1				Prior2			
							I.T - SE		MCMC - SE		I.T - SE		MCMC - SE	
			EL	CP	EL	CP	EL	CP	EL	CP	EL	CP	EL	CP
(10,10)	0.3032	(r_1,r_2)	0.3830	0.899	0.3700	0.939	0.3753	0.920	0.3774	0.929	0.3611	0.941	0.3630	0.939
		(r_1,r_3)	0.3836	0.891	0.3770	0.928	0.3851	0.948	0.3838	0.931	0.3715	0.926	0.3719	0.940
		(r_2,r_3)	0.3834	0.903	0.3756	0.929	0.3841	0.944	0.3840	0.955	0.3708	0.958	0.3713	0.939
		(r_1,r_1)	0.4343	0.931	0.3916	0.925	0.3715	0.941	0.3734	0.960	0.3603	0.965	0.3596	0.963
	0.5	(r_2,r_2)	0.4342	0.923	0.3948	0.936	0.3723	0.944	0.3721	0.948	0.3591	0.937	0.3600	0.954
		(r_3,r_3)	0.4339	0.923	0.4222	0.955	0.3722	0.950	0.3726	0.943	0.3600	0.960	0.3593	0.950
		(r_1,r_2)	0.4351	0.937	0.3937	0.940	0.3576	0.934	0.3605	0.941	0.3494	0.943	0.3501	0.939
		(r_1,r_3)	0.4345	0.933	0.4074	0.949	0.2963	0.900	0.2936	0.902	0.3036	0.909	0.3030	0.912
		(r_2,r_3)	0.4336	0.928	0.4078	0.949	0.3310	0.910	0.3281	0.914	0.3268	0.922	0.3295	0.931
	0.6967	(r_1,r_1)	0.3827	0.896	0.3665	0.925	0.3556	0.940	0.3543	0.948	0.3440	0.961	0.3433	0.953
		(r_2,r_2)	0.3818	0.900	0.3684	0.923	0.3560	0.944	0.3560	0.949	0.3428	0.959	0.3427	0.957
		(r_3,r_3)	0.3828	0.892	0.3876	0.935	0.3545	0.946	0.3558	0.941	0.3408	0.944	0.3452	0.969
		(r_1,r_2)	0.3836	0.901	0.3665	0.922	0.2898	0.923	0.2929	0.923	0.2871	0.943	0.2902	0.937
		(r_1,r_3)	0.3825	0.896	0.3792	0.937	0.1401	0.869	0.1456	0.878	0.1725	0.898	0.1751	0.901
		(r_2,r_3)	0.3795	0.889	0.3801	0.938	0.2133	0.895	0.2178	0.879	0.2275	0.891	0.2302	0.899
(15,15)	0.3032	(r_1,r_1)	0.3254	0.917	0.3145	0.944	0.3038	0.931	0.3052	0.935	0.2962	0.948	0.2974	0.949
		(r_2,r_2)	0.3253	0.913	0.3143	0.943	0.3029	0.933	0.3035	0.947	0.2963	0.953	0.2981	0.960
		(r_3,r_3)	0.3228	0.906	0.3274	0.950	0.3028	0.929	0.3022	0944	0.2954	0.950	0.2973	0.960
		(r_1,r_2)	0.3245	0.912	0.3130	0.928	0.3430	0.922	0.3275	0.923	0.3125	0.937	0.3153	0.930
		(r_1,r_3)	0.3248	0.921	0.3176	0.928	0.3246	0.920	0.3262	0.921	0.3187	0.928	0.3194	0.930
		(r_2,r_3)	0.3235	0.905	0.3188	0.936	0.3286	0.918	0.3286	0.921	0.3199	0.929	0.3201	0.932
	0.5	(r_1,r_1)	0.3548	0.933	0.3317	0.949	0.3174	0.947	0.3172	0.938	0.3094	0.946	0.3096	0.950
		(r_2,r_2)	0.3547	0.933	0.3339	0.959	0.3168	0.938	0.3172	0.940	0.3102	0.964	0.3103	0.957
		(r_3,r_3)	0.3551	0.940	0.3484	0.963	0.3183	0.952	0.3169	0.947	0.3098	0.945	0.3105	0.957
		(r_1,r_2)	0.3550	0.937	0.3324	0.933	0.3016	0.915	0.3038	0.911	0.2961	0.926	0.2976	0.921
		(r_1,r_3)	0.3555	0.938	0.3389	0.942	0.2307	0.878	0.2312	0.883	0.2409	0.891	0.2427	0.897
		(r_2,r_3)	0.3554	0.932	0.3405	0.945	0.2685	0.898	0.2723	0.890	0.2707	0.901	0.2700	0.892
	0.6967	(r_1,r_1)	0.3208	0.902	0.3109	0.936	0.3040	0.940	0.3018	0.931	0.2961	0.952	0.2959	0.957
		(r_2,r_2)	0.3220	0.913	0.3144	0.951	0.3030	0.945	0.3045	0.944	0.2965	0.942	0.2974	0.959
		(r_3,r_3)	0.3256	0.930	0.3245	0.945	0.3025	0.929	0.3049	0.942	0.2964	0.948	0.2973	0.957
		(r_1,r_2)	0.3217	0.909	0.3128	0.929	0.2362	0.902	0.2393	0.906	0.2397	0.908	0.2389	0.910
		(r_1,r_3)	0.3231	0.917	0.3187	0.935	0.0939	0.846	0.0953	0.839	0.1165	0.861	0.1181	0.869
		(r_2,r_3)	0.3232	0.901	0.3188	0.925	0.1637	0.878	0.1632	0.869	0.1728	0.876	0.1756	0.878

Table 5. (continued)

(m_1,m_2)	R	C.S	GPQ		Bayes				Bayes			
					Prior1				Prior2			
					MCMC - LINEX		MCMC - Stein		MCMC - LINEX		MCMC - Stein	
			EL	CP	EL	CP	EL	CP	EL	CP	EL	CP
(5,5)	0.3032	(r_1,r_1)	0.5434	0.956	0.4495	0.966	0.4473	0.959	0.4296	0.984	0.4290	0.989
		(r_2,r_2)	0.5370	0.961	0.4497	0.972	0.4502	0.969	0.4269	0.982	0.4294	0.982
		(r_3,r_3)	0.5353	0.960	0.4469	0.970	0.4466	0.959	0.4279	0.989	0.4277	0.981
		(r_1,r_2)	0.5332	0.950	0.4685	0.960	0.4665	0.960	0.4456	0.978	0.4461	0.984
		(r_1,r_3)	0.5419	0.965	0.4850	0.920	0.4858	0.912	0.4604	0.937	0.4583	0.960
		(r_2,r_3)	0.5392	0.967	0.4807	931 0	0.4798	0.942	0.4542	0.932	0.4545	0.959
	0.5	(r_1,r_1)	0.5441	0.959	0.4720	0.962	0.4721	0.962	0.4482	0.978	0.4506	0.983
		(r_2,r_2)	0.5457	0.947	0.4716	0.972	0.4735	0.964	0.4493	0.984	0.4501	0.977
		(r_3,r_3)	0.5409	0.940	0.4741	0.974	0.4738	0.963	0.4507	0.988	0.4491	0.980
		(r_1,r_2)	0.5450	0.949	0.4672	0.947	0.4666	0.932	0.4455	0.951	0.4458	0.971
		(r_1,r_3)	0.5451	0.956	0.4316	0.932	0.4319	0.924	0.4255	0.937	0.4274	0.942
		(r_2,r_3)	0.5477	0.959	0.4500	0.942	0.4516	0.943	0.4361	0.945	0.4359	0.941
	0.6967	(r_1,r_1)	0.5332	0.960	0.4488	0.961	0.4474	0.966	0.4256	0.975	0.4282	0.984
		(r_2,r_2)	0.5355	0.956	0.4467	0.963	0.4489	0.968	0.4268	0.989	0.4273	0.982
		(r_3,r_3)	0.5384	0.961	0.4493	0.978	0.4494	0.970	0.4291	0.989	0.4276	0.980
		(r_1,r_2)	0.5344	0.948	0.3957	0.915	0.3960	0.919	0.3879	0.927	0.3877	0.931
		(r_1,r_3)	0.5350	0.958	0.2903	0.912	0.2901	0.916	0.3155	0.910	0.3193	0.912
		(r_2,r_3)	0.5378	0.954	0.3408	0.925	0.3387	0.922	0.3525	0.931	0.3491	0.929

Table 5. (continued)

(m_1,m_2)	R	C.S	GPQ		Bayes				Bayes			
					Prior1				Prior2			
			EL	CP	MCMC - LINEX		MCMC - Stein		MCMC - LINEX		MCMC - Stein	
					EL	CP	EL	CP	EL	CP	EL	CP
(10,10)	0.3032	(r_1, r_1)	0.3964	0.938	0.3572	0.958	0.3589	0.959	0.3431	0.959	0.3444	0.952
		(r_2, r_2)	0.3973	0.951	0.3578	0.953	0.3559	0.949	0.3455	0.965	0.3437	0.962
		(r_3, r_3)	0.3963	0.955	0.3545	0.933	0.3573	0.946	0.3455	0.964	0.3449	0.959
		(r_1, r_2)	0.4004	0.954	0.3766	0.943	0.3755	0.915	0.3624	0.946	0.3621	0.948
		(r_1, r_3)	0.3963	0.949	0.3852	0.944	0.3840	0.948	0.3723	0.944	0.3733	0.954
		(r_2, r_3)	0.3947	0.952	0.3857	0.940	0.3843	0.934	0.3713	0.951	0.3717	0.941
	0.5	(r_1, r_1)	0.4075	0.948	0.3722	0.950	0.3721	0.953	0.3595	0.964	0.3594	0.953
		(r_2, r_2)	0.4089	0.943	0.3722	0.966	0.3723	0.946	0.3593	0.967	0.3602	0.965
		(r_3, r_3)	0.4088	0.956	0.3711	0.941	0.3719	0.948	0.3597	0.965	0.3598	0.950
		(r_1, r_2)	0.4094	0.958	0.3581	0.930	0.3582	0.929	0.3500	0.948	0.3499	0.944
		(r_1, r_3)	0.4074	0.945	0.2976	0.906	0.2927	0.902	0.3037	0.911	0.3049	0.910
		(r_2, r_3)	0.4081	0.954	0.3274	0.923	0.3316	0.913	0.3280	0.928	0.3282	0.931
	0.6967	(r_1, r_1)	0.3973	0.954	0.3551	0.939	0.3576	0.952	0.3452	0.962	0.3430	0.949
		(r_2, r_2)	0.3984	0.956	0.3554	0.960	0.3566	0.948	0.3429	0.970	0.3454	0.958
		(r_3, r_3)	0.3983	0.955	0.3560	0.940	0.3548	0.939	0.3432	0.953	0.3457	0.971
		(r_1, r_2)	0.3955	0.952	0.2906	0.912	0.2919	0.916	0.2881	0.917	0.2913	0.928
		(r_1, r_3)	0.3961	0.948	0.1472	0.899	0.1439	0.874	0.1768	0.901	0.1758	0.893
		(r_2, r_3)	0.3992	0.952	0.2096	0.904	0.2168	0.903	0.2292	0.916	0.2303	0.909
(15,15)	0.3032	(r_1, r_1)	0.3284	0.951	0.3039	0.935	0.3058	0.948	0.2974	0.947	0.2969	0.959
		(r_2, r_2)	0.3300	0.954	0.3027	0.933	0.3051	0.944	0.2966	0.954	0.2963	0.952
		(r_3, r_3)	0.3298	0.960	0.3054	0.943	0.3027	0.938	0.2967	0.955	0.2953	0.941
		(r_1, r_2)	0.3281	0.945	0.3237	0.932	0.3222	0.931	0.3143	0.937	0.3151	0.935
		(r_1, r_3)	0.3301	0.963	0.3257	0.928	0.3259	0.925	0.3187	0.922	0.3191	0.921
		(r_2, r_3)	0.3309	0.950	0.3282	0.929	0.3285	0.930	0.3201	0.934	0.3202	0.928
	0.5	(r_1, r_1)	0.3409	0.949	0.3172	0.940	0.3183	0.948	0.3087	0.952	0.3096	0.959
		(r_2, r_2)	0.3409	0.956	0.3171	0.934	0.3179	0.951	0.3090	0.960	0.3095	0.946
		(r_3, r_3)	0.3404	0.960	0.3167	0.937	0.3178	0.942	0.3098	0.955	0.3099	0.960
		(r_1, r_2)	0.3402	0.945	0.3011	0.921	0.3222	0.930	0.2985	0.927	0.2971	0.929
		(r_1, r_3)	0.3411	0.961	0.2302	0.909	0.2317	0.905	0.2398	0.906	0.2390	0.901
		(r_2, r_3)	0.3406	0.945	0.2697	0.918	0.2716	0.920	0.2707	0.919	0.2715	0.919
	0.6967	(r_1, r_1)	0.3316	0.955	0.3039	0.937	0.3042	0.939	0.2951	0.945	0.2967	0.954
		(r_2, r_2)	0.3307	0.952	0.3052	0.937	0.3041	0.938	0.2967	0.962	0.2969	0.947
		(r_3, r_3)	0.3302	0.955	0.3033	0.946	0.3032	0.957	0.2968	0.960	0.2981	0.955
		(r_1, r_2)	0.3307	0.946	0.2408	0.920	0.2385	0.911	0.2405	0.912	0.2381	0.915
		(r_1, r_3)	0.3312	0.955	0.0940	0.858	0.0934	0.843	0.1160	0.876	0.1177	0.882
		(r_2, r_3)	0.3292	0.947	0.1638	0.908	0.1629	0.903	0.1751	0.911	0.1753	0.902

The following results could be obtained from Table 5:

1- The EL of CIs based on GPQ (bootstrap based on ML estimators) is wider than the other CIs when the parameter R is not close (close) to 0.5. Also, the EL of all CIs maximized at R=0.5. Furthermore, the HPD credible intervals have shorter length than the other CIs.

2- Comparing the non-Bayesian CIs, for large sample sizes, the EL of CI based on BLU estimates is shorter than the bootstrap CIs based on the ML estimates and GPQ.

3- For fixed censoring scheme, the EL of all CIs decreases as sample size increases.

4- The simulation results show that the GCIs provide proper CP. Also, as sample size increases, the coverage probability of bootstrap CIs get close to 95%.

A Real Data Analysis

We apply our proposed estimation methods to analyze two real datasets. These datasets consist of breaking strengths of jute fibers at two different gauge lengths, as previously utilized by ([25]). The data in Tables 6 and 7 represent the breaking strength of jute fibers at lengths of 10mm and 20mm.

First, we fit the two-parameter exponential distribution to the two data sets separately. The ML estimates of μ_1, σ_1 and μ_2, σ_2 are 43.93, 321.80, 36.75 and 303.99 respectively. The Kolmogorov-Smirnov (K-S) distances between the empirical distribution functions and the fitted distribution functions have been used to check the goodness-of-fit. The (K-S) test statistics are 0.1183 and 0.1469 and the associated p values are 0.7517 and 0.4908, respectively. Based on the p values, one cannot reject the hypothesis that the data are coming from two-parameter exponential distributions. Moreover, we plot the empirical distribution functions and the fitted distribution functions in Figure 1.

The generated data and corresponding censored schemes have been shown in Table 8.

The bootstrap percentile interval is calculated based on 10000 parametric bootstrap resamples using the ML and BLU estimates. The interval endpoints are found as the lower and upper 2.5% quantiles of the estimated bootstrap distribution of R. The ML and BLU estimates of R become $\hat{R} = 0.5167$, $\tilde{R} = 0.5135$. The corresponding 95% confidence intervals from (8)-(10) based on ML estimates are equal to (0.3152, 0.7182), (0.3088, 0.7201), (0.3460, 0.5694), and based on BLU estimates are equal to (0.3029, 0.7241), (0.3199 0.7019), (0.3391, 0.5869). The Bayesian intervals are obtained based on two procedure generating μ_i, which are

Table 6. Data Set 1

43.93	50.16	101.15	108.94	123.06	141.38	151.48	163.40	177.25	183.16
212.13	257.44	262.90	291.27	303.90	323.83	353.24	376.42	383.43	422.11
506.60	530.55	590.48	637.66	671.49	693.73	700.74	704.66	727.23	778.18

Table 7. Data Set 2

36.75	45.58	48.01	71.46	83.55	99.72	113.85	116.99	119.86	145.96
166.49	187.13	187.85	200.16	244.53	284.64	350.70	375.81	419.02	456.60
547.44	578.62	581.60	585.57	594.29	662.66	688.16	707.36	756.70	765.14

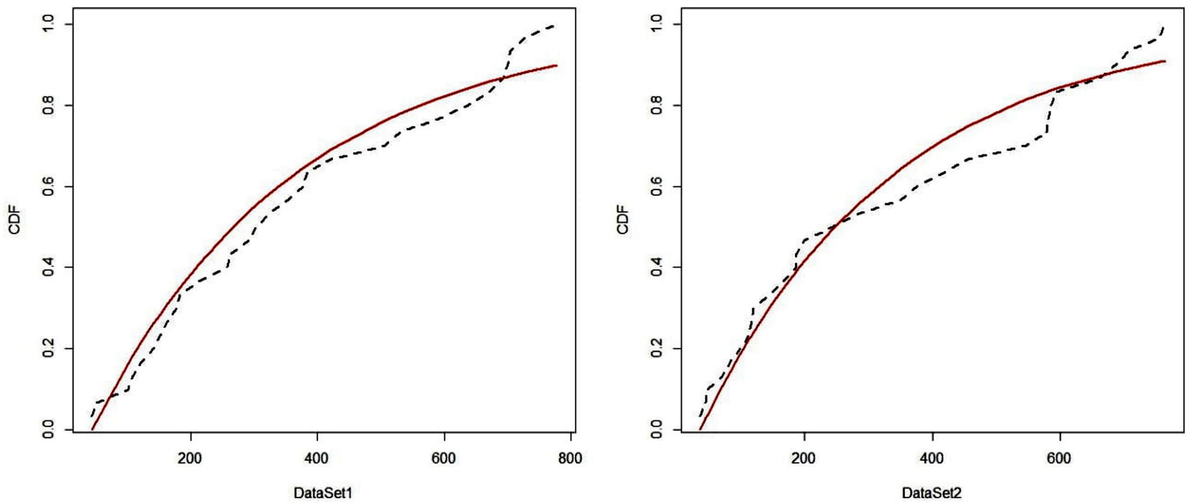

Figure 1. The empirical distribution function (dashed) and fitted distribution functions for Data Sets 1 and 2.

Table 8. Data and the corresponding censored schemes

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	43.93	108.94	141.38	183.16	212.13	303.90	353.24	422.11	506.60	590.48	693.73	700.74
R_i	2	1	3	0	3	1	2	0	1	2	0	3
y_i	36.75	48.01	99.72	113.85	145.96	187.13	284.64	350.70	547.44	581.60	688.16	756.70
$R_i^{'}$	1	2	0	2	1	3	0	3	1	3	1	1

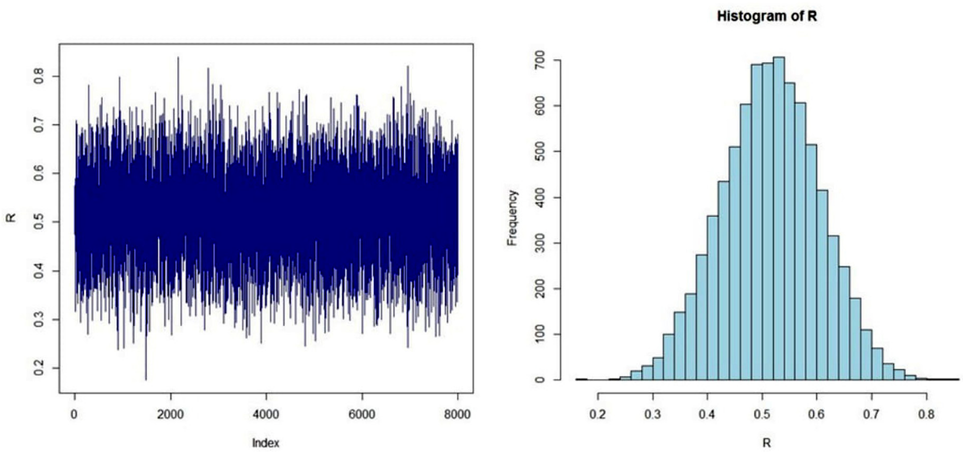

Figure 2. Simulated values of *R* and histogram of *R*.

named inverse transformation algorithm and MCMC. We obtained 10000 observations from the posterior distribution of R and computed the Bayes estimates under SE, LINEX with *b=1* and Stein loss functions. The HPD interval based on prior 2 is found using the algorithm of Chen and Shao.[21] The approximate Bayes estimate of R under the SE loss function by inverse transformation algorithm using (5) based on *B=10000* samples and discard the first *M=2000* values as burn-in period becomes $\hat{R}_{BS} = 0.5162$, and by MCMC becomes $\hat{R}_{BS} = 0.5193$. From (6), the approximate Bayes estimate of R under the LINEX loss function is given as $\hat{R}_{BL} = 0.5067$. Also under the Stein loss function, the Bayes estimate of R is computed as $\hat{R}_{BST} = 0.5005$. Furthermore, using the method of Chen and Shao,[21] the 95% HPD credible intervals of R under SE loss function by inverse transformation algorithm and MCMC, LINEX and Stein loss functions are obtained as (0.3462, 0.6951), (0.3497, 0.6978), (0.3418, 0.6917) and (0.3396, 06881), respectively. The simulated values and histogram of R generated by the algorithm of MCMC are plotted in Figure 2. In this example, the scale factor value of the MCMC estimates based on 20 sequences is 0.999 which is an acceptable value for their convergence. It can be seen that the Boot-t confidence interval has the shortest expected length. All intervals cover the exact value of the stress-strength reliability parameter.

Discussion

In this article, we proposed point and interval estimation of the stress-strength reliability parameter under censored samples. Simulation results show that the Bayes estimator under the LINEX loss function is better than the other estimators in terms of ER, and the HPD credible intervals (GCIs) are better than the other CIs in terms of EL (CP).

Conclusions

Based on the progressive type-II censored samples, the point estimates and confidence intervals of the stress-strength reliability $R=P(Y<X)$ investigated, where X and Y are two random variables from two-parameter exponential distribution with unknown parameters. The results demonstrated that with increasing the sample size, in almost cases the ERs of all the estimators decrease. Also, in almost all cases the Bayes estimator under the LINEX loss function has smaller ER than the other estimators. Furthermore, the Bayes estimators are sensitive to the values of the parameters of prior distributions.

Based on our simulation, the ELs of all intervals tend to decrease when the sample size increases. Also the HPD confidence intervals are shorter than the others intervals for all the values of R. Moreover, the CP of GCIs are near the nominal level for different sample sizes and values of R.

Conflict of interest

The author declares that there are no conflicts of interest related to this paper.

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