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Original Article

A new transformed Weibull lifetime distribution and its inferences based on the Bayes and maximum likelihood procedures

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ABSTRACT

Background & Aim: In last three decades or so, an extensive research works has appeared in the literature on the theory of statistical distributions. The Weibull distribution is a very popular model, and has been extensively used over the past decades for modelling data in reliability, engineering and biological studies.
Methods & Materials: First, we obtain some of important statistical and reliability characteristics of the new model, and then the estimation of the parameters of proposed model is studied through two views of Bayesian and classic statistics.
Results: We show that the new distribution has the ability to fit into complete and censored real data. In the application section, we show the superiority of the proposed model to some common statistical distributions.
Conclusion: In this paper, we have proposed a new transformed Weibull distribution, denoted by TWD. It is investigated that the new model has increasing, decreasing and bathtub shape hazard functions. We provide the comprehensive Bayesian and maximum likelihood estimation procedures for complete and right censored real observations.

Introduction

In last three decades or so, an extensive research works has appeared in the literature on the theory of statistical distributions. The Weibull distribution is a very popular model, and has been extensively used over the past decades for modelling data in reliability, engineering and biological studies. Motivated by engineering applications, Weibull (1939), a Swedish physicist, suggested a distribution that has proved to be of seminal importance in reliability. The corresponding survival function is given by the equation

$$\bar{F}(x) = \exp\left(-\lambda x^{\beta}\right) ,x > 0 ,$$

with parameters $\beta , \lambda > 0$.

For many researchers, the Weibull distribution is of great importance, and therefore various generalizations of this distribution are presented.

In the present paper, we introduce a new model based on The Weibull Distribution (TWD) and provide a comprehensive description of some mathematical properties with the hope that it will attract wider applications in reliability, engineering and in other areas of research. In addition, we estimate the parameters of proposed model from two views of classic and Bayesian inferential statistics. The interesting TWD distribution has several desirable properties, especially it has closed relations with Weighted Exponential (WE) and Weighted Weibull (WW) distributions. The class of weighted exponential distribution was introduced in the seminal paper by Gupta and Kundu (2009) and have received a great deal of attention in recent years. Weighted exponential distribution denoted by $WE(\lambda, \alpha)$ has probability density function with PDF

$$f_X(x, \alpha, \lambda) = \frac{\alpha+1}{\alpha} \lambda e^{-\lambda x}\left(1 - e^{-\lambda \alpha x}\right), \quad (1)$$

where $x > 0$, $\alpha > 0$ and $\lambda > 0$. Here α

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and λ are the shape and scale parameters, respectively.

A random variable X is said to have weighted Weibull distribution, denoted by $WW(\alpha,\beta,\lambda)$, if its probability density function (PDF) is given as

$$f_X(x,\alpha,\beta,\lambda) = \frac{\alpha+1}{\alpha}\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(1-e^{-\lambda\alpha x^{\beta}}\right), \qquad (2)$$

where $x>0$, $\alpha>0$, $\beta>0$ and $\lambda>0$. Here α,β and λ are the shape and scale parameters, respectively. See Kharazmi (2016).

Our new proposed model provides more flexibility to fitting censored and uncensored survival data in the real applications. For illustrative purposes we use two real data sets, and it is observed that TWD provides better fit than WE model and Weibull distributions.

Methods

In this section, we introduce the definition of the new transformed Weibull distribution denoted by $TWD(\alpha,\beta,\lambda)$ and also two stochastic representations are given here.

Definition 1. A random variable X is said to have a new transformed Weibull distribution $TWD(\alpha,\beta,\lambda)$, with shape parameters $\alpha>0,\beta>0$ and scale parameter $\lambda>0$, if the PDF of X is given as following

$$f_X(x,\alpha,\beta,\lambda) = \frac{\alpha+1}{\alpha}\lambda^2\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(x^{\beta}-\frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x^{\beta}}\right)\right), \alpha>0,\ \beta,\ \lambda>0, x>0. \qquad (3)$$

Figure 1 shows the PDF of the TWD distribution for selected values of parameters.

In the next, we explore the relation between proposed model (3) with WE and WW models considered in (1) and (2), respectively. Representation1 shows the connection between

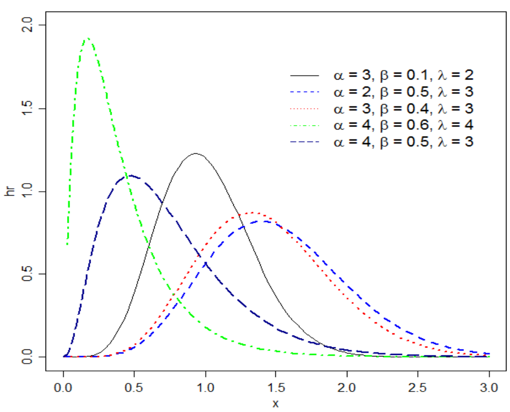

Figure 1. Plots of the PDF function of the TWD distribution for some selected values of parameters.

proposed model and WE distribution and representation 2 shows the connection between proposed model and WW distribution.

Representation 1. Suppose that U and V be two independent random variables as $U\sim$ WE(α,λ) and $V\sim\exp(\lambda)$ then the transformed variable $X=\sqrt[\beta]{(U+V)}$,

has the PDF with (3).

Representation 2. The TWD distribution can be stated as mixtures of weighted Weibull distribution and length biased Weibull as following

$$f_X(x,\alpha,\beta,\lambda) = \lambda f_{X_1}(x,\alpha,\beta,\lambda) + (1-\lambda)f_{X_2}(x,\alpha,\beta,\lambda), \qquad (4)$$

where $\lambda=\frac{\alpha+1}{\alpha}$ and X_1 and X_2 have length biased Weibull $LBW(\alpha,\beta,\lambda)$ an weighted Weibull $WW(\alpha,\beta,\lambda)$, respectively.

Remark. Both above stochastic representations can be used to generate random sample from TWD distribution. Note that the simplest way to generate WTD random number is to use the stochastic representation 1.

In the next, we obtain the CDF of TWD distribution based on the representation 2 as

$$F_X(x,\alpha,\beta,\lambda) = \lambda F_1(x,\beta,\lambda) + (1-\lambda)F_2(x,\alpha,\beta,\lambda)$$

where $F_1(x,\beta,\lambda)$ is

$$F_{X_1}(x,\beta,\lambda) = 1-e^{-\lambda x^{\beta}}-x^{\beta}e^{-\lambda x^{\beta}}$$

and $F_2(x,\alpha,\beta,\lambda)$ is

$$F_{X_2}(x,\alpha,\beta,\lambda) = \frac{\alpha+1}{\alpha}(1-e^{-\lambda x^{\beta}}-\frac{1}{\alpha+1}(1-e^{-\lambda(\alpha+1)x^{\beta}})).$$

In the next, we obtain the survival function (SF), moment generating function (MGF), hazard function (HF) and order statistics of proposed model that is given in (3). Some of the most important features and characteristics of a distribution can be studied through its moment generating function. The moment generating function of (3) is immediately written as,

$$M_X(t) = E(e^{tx}) = \lambda M_T(t) + (1-\lambda)M_Z(t),$$

where $T\sim LBW(\alpha,\beta,\lambda)$ and $Z\sim WW(\alpha,\beta,\lambda)$

and $M_{T^\beta}(t) = (\frac{\lambda}{\lambda-t})^2$ and where $t \in \{t \mid M_X(t) < \infty\}$.

Now, we can obtain the expectation of X as

$$(X) = \lambda^2 E(X_1{}^\beta) + (1-\lambda)E(X_2),$$

$$M_Z(t) = \frac{\alpha^\beta}{B(1/\alpha^\beta, 2)}\sum_{i=0}^{\infty}\sum_{j=0}^{1}(-1)^j \frac{t^i}{i!}\binom{1}{j}\lambda\frac{\Gamma\left(\frac{i}{\beta}+1\right)}{(\lambda(\alpha^\beta j+1))^{\frac{i}{\beta}+1}}$$

Where $X_1 \sim Weibull(\beta, \lambda)$ and $X_2 \sim WW(\alpha, \beta, \lambda)$.

Survival function: The survival function of the TWD distribution is given by

$$\overline{F}_X(x,\alpha,\beta,\lambda) = \lambda\overline{F}_1(x,\beta,\lambda) + (1-\lambda)\overline{F}_2(x,\alpha,\beta,\lambda),$$

Where

$$\overline{F}_{X_1}(x,\beta,\lambda) = e^{-\lambda x^\beta} + x^\beta e^{-\lambda x^\beta}$$

And

$$\overline{F}_{X_2}(x,\alpha,\beta,\lambda) = 1 - \frac{\alpha+1}{\alpha}(1-e^{-\lambda x^\beta} - \frac{1}{\alpha+1}(1-e^{-\lambda(\alpha+1)x^\beta})).$$

The hazard function (HF) of X can be written as

$$Hr(x,\lambda) == \frac{\lambda f_T(x) + (1-\lambda)Ff_Z(x)}{1-[\lambda F_T(x) + (1-\lambda)F_Z(x)]},$$

$T^\beta \sim Gamma(2,\lambda)$and $Z \sim WW(\alpha,\beta,\lambda)$.

The hazard rate function allows for monotonically increasing, monotonically decreasing and upside bathtub shaped hazard rates. In Figures 2, 3 and 4, we plotted the hazard rate function of the TWD distribution in three cases for selected values of parameters.

Here we provide an order statistics result. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $TWD(\alpha,\beta,\lambda)$ and let $X_{i:n}$ denote the ith order statistic.

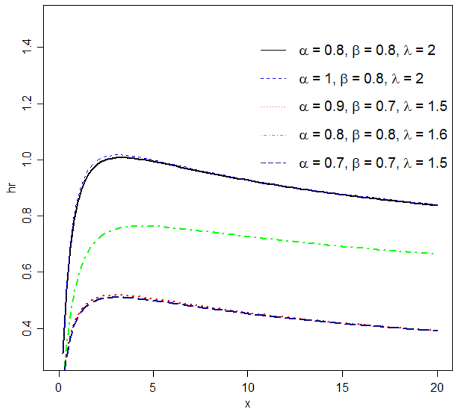

Figure 2. Plots of the hazard rate function of the TWD distribution for some selected values of parameters (upside-down bathtub shape).

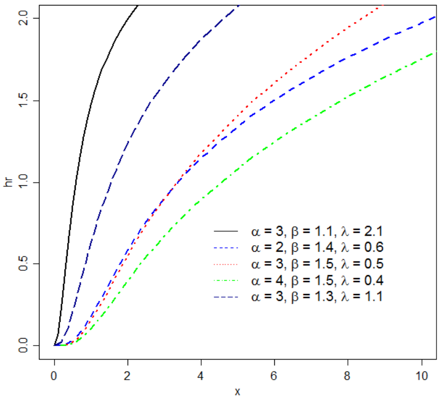

Figure 3. Plots of the hazard rate function of the TWD distribution for some selected values of parameters (increasing shape).

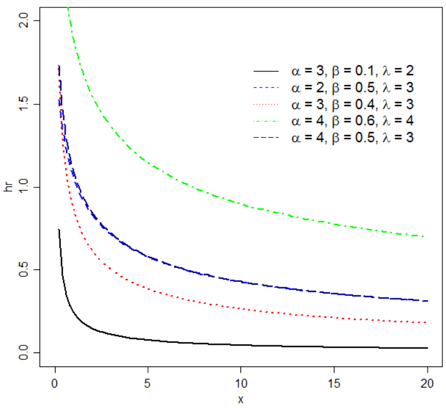

Figure 4. Plots of the hazard rate function of the TWD distribution for some selected values of parameters (decreasing shape).

The PDF of $X_{i:n}$ is given by

$$f_{x(i)}(x) = \frac{n!}{(i-1)!\,(n-i)!}\frac{\alpha+1}{\alpha}\lambda^2\beta x^{\beta-1}e^{-\lambda x^\beta}\left(x^\beta - \frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x^\beta}\right)\right)\times$$

$$\left(\lambda F_1(x,\alpha,\beta,\lambda)+(1-\lambda)F_2(x,\alpha,\beta,\lambda)\right)^{i-1}(1-\lambda F_1(x,\alpha,\beta,\lambda)-(1-\lambda)F_2(x,\alpha,\beta,\lambda))^{n-i}$$

In the next section, we consider both classic and Bayesian inferences for the parameters of TWD distribution for complet and censored data setting.

Estimation: In this section, we describe two well-known estimation methods, Bayesian and maximum likelihood procedures, that considered in this paper for estimating the parameters α, β and λ of the TWD distribution. In addition, these methods are used for complete and right censored observations. We consider the case when all three parameters α, β and λ are unknown.

Maximum likelihood estimation: The maximum likelihood procedure is one of the most common methods for obtaining an estimator for a unknown parameter in classic statistical inference. The likelihood function is a function that is written based on the mechanism of the occurrence of observations.

Complete maximum likelihood: We obtain the normal equations for finding the maximum likelihood estimators (MLEs) of parameters in complete data setting.

Suppose $X_1,\dots,X_n$ be a random sample from $TWD(\alpha,\beta,\lambda)$. The log-likelihood function based on the observed sample $(x_1,\dots,x_n)$ is

$$l(\boldsymbol{\theta}) = \ln L(x_1,\dots,x_n|\boldsymbol{\theta})$$
$$= n(\log(\alpha+1) - \log\alpha + \log\beta + 2\log\lambda),$$
$$-\lambda\sum_{i=1}^{n} x_i{}^\beta + \sum_{i=1}^{n}\log\left(x_i{}^\beta - \frac{1}{\lambda\alpha}\left(1-e^{-\lambda\alpha x_i{}^\beta}\right)\right)$$ where $\boldsymbol{\theta} = (\alpha,\beta,\lambda)$.

To find the MLE estimates for the TWD model parameters, we differentiate the log-likelihood function and equating the resulting to 0 as follows

$$\frac{\partial L}{\partial\alpha} = n\left(\frac{1}{\alpha+1}-\frac{1}{\alpha}\right) + \sum_{i=1}^{n}\left[\frac{\frac{1}{\alpha^2\lambda}-\left(\frac{1}{\lambda\alpha^2}+\frac{x_i{}^\beta}{\alpha}\right)e^{-\lambda\alpha x_i{}^\beta}}{x_i{}^\beta - \frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x_i{}^\beta}\right)}\right] = 0$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} - \lambda\ln x_i\sum_{i=1}^{n}x_i{}^\beta + \sum_{i=1}^{n}\left[\frac{(1-e^{-\lambda\alpha x_i{}^\beta})x_i{}^\beta\ln x_i}{x_i{}^\beta-\frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x_i{}^\beta}\right)}\right] = 0$$

$$\frac{\partial L}{\partial\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i{}^\beta + \sum_{i=1}^{n}\frac{\frac{1}{\alpha\lambda^2}-\left(\left(\frac{1}{\alpha\lambda^2}+\frac{x_i{}^\beta}{\lambda}\right)e^{-\lambda\alpha x_i{}^\beta}\right)}{x_i{}^\beta-\frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x_i{}^\beta}\right)} = 0.$$

The MLEs of the unknown parameters cannot be obtained explicitly. They have to be obtained by solving some numerical methods, like Newton-Raphson or Gauss-Newton methods or their variants.

Censored maximum-likelihood: In real life, sometimes it is hard to get a complete data set. Often with lifetime data, one encounters censoring. There are different forms of censoring: type I, type II, etc. Here, we consider the type II (right) censored data, the likelihood function based on a sample size n is given as

$$L(\underline{x},\underline{\delta},\underline{\theta}) = \prod_{i=1}^{n}(f(x_i,\underline{\theta}))^{\delta_i}\left(1-F(x_i,\underline{\theta})\right)^{1-\delta_i},$$

where δ_i is a censoring indicator variable, that is, $\delta_i = 1$ for an observed survival time and $\delta_i = 0$ for a right-censored survival time. In the case TWD distribution the likelihood function and the corresponding log-likelihood are given as

$$L(\underline{x},\underline{\delta},\alpha,\beta,\lambda) = \prod_{i=1}^{n}\left(\frac{\alpha+1}{\alpha}\lambda^2\beta x_i{}^{\beta-1}e^{-\lambda x_i{}^\beta}\left(x_i{}^\beta-\frac{1}{\alpha\lambda}\left(1-e^{-\lambda\alpha x_i{}^\beta}\right)\right)\right)^{\delta_i}\left(1-wF_1(x_i)+(1-w)F_2(x_i)\right)^{1-\delta_i}$$

and

$$\ell = \operatorname{Log} L(\underline{X}, \underline{\delta}, \alpha, \beta, \lambda) = \sum_{i=1}^{n} \delta_i \log(\frac{\alpha+1}{\alpha} \lambda^2 \beta {x_i}^{\beta-1} e^{-\lambda {x_i}^{\beta}} \left({x_i}^{\beta} - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha {x_i}^{\beta}}\right)\right))$$

$$+\sum_{i=1}^{n}(1-\delta_i)\log(1 - wF_1(x_i) - (1-w)F_2(x_i)) = \ell_1 + \ell_2$$

AND

$$\ell = \operatorname{Log} L(\underline{X}, \underline{\delta}, \alpha, \beta, \lambda) = \sum_{i=1}^{n} \delta_i \log(\frac{\alpha+1}{\alpha} \lambda^2 \beta {x_i}^{\beta-1} e^{-\lambda {x_i}^{\beta}} \left({x_i}^{\beta} - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha {x_i}^{\beta}}\right)\right))$$

$$+\sum_{i=1}^{n}(1-\delta_i)\log(1 - wF_1(x_i) - (1-w)F_2(x_i)) = \ell_1 + \ell_2$$

where

$F_2(x) = \frac{\alpha+1}{\alpha}(1 - e^{-\lambda x^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1)x^{\beta}}\right))$ $F_{X_1}(x) = 1 - e^{-\lambda x^{\beta}} - x^{\beta} e^{-\lambda x^{\beta}}$,
and $w = \frac{\alpha+1}{\alpha}$.

The normal equations are provided as

$\frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell_1}{\partial \alpha} + \frac{\partial \ell_2}{\partial \alpha} = 0$

where

$\frac{\partial \ell_1}{\partial \alpha} = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha+1} - \frac{1}{\alpha} + \frac{\frac{1}{\alpha^2\lambda}(1 - e^{-\lambda {x_i}^{\beta}})}{{x_i}^{\beta} - \frac{1}{\alpha\lambda}(1 - e^{-\lambda {x_i}^{\beta}})}\right]$ and

$$\frac{\partial \ell_2}{\partial \alpha} = \sum_{i=1}^{n}(1-\delta_i) \times$$

$$\left[\frac{\frac{1}{\alpha^2}\left(1 - e^{-\lambda {x_i}^{\beta}} - {x_i}^{\beta} e^{-\lambda {x_i}^{\beta}}\right) + (\frac{\alpha^2+2\alpha}{\alpha^4})(1 - e^{-\lambda {x_i}^{\beta}}) - \frac{2\alpha}{\alpha^4}\left(1 - e^{-\lambda(\alpha+1){x_i}^{\beta}}\right)}{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-\lambda {x_i}^{\beta}} - {x_i}^{\beta} e^{-\lambda {x_i}^{\beta}}\right) - \frac{1}{\alpha}[\frac{\alpha+1}{\alpha}\left(1 - e^{-\lambda {x_i}^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1){x_i}^{\beta}}\right)\right)]}\right] +$$

$$\sum_{i=1}^{n}(1 - \delta_i)\left[\frac{\frac{1}{\alpha^2}\left(\lambda {x_i}^{\beta} e^{-\lambda(\alpha+1){x_i}^{\beta}}\right)}{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-\lambda {x_i}^{\beta}} - {x_i}^{\beta} e^{-\lambda {x_i}^{\beta}}\right) - \frac{1}{\alpha}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-\lambda {x_i}^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1){x_i}^{\beta}}\right)\right)\right]}\right].$$

$$\frac{\partial \ell}{\partial \beta} = \frac{\partial \ell_1}{\partial \beta} + \frac{\partial \ell_2}{\partial \beta} = 0$$

where

$$\frac{\partial \ell_1}{\partial \beta} = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\beta} + \log x_i - \lambda x_i{}^{\beta} \ln x_i + \frac{x_i{}^{\beta} \ln x_i - \left(\frac{x_i{}^{\beta}}{\alpha} e^{-\lambda x_i{}^{\beta}} \ln x_i \right)}{x_i{}^{\beta} - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda x_i{}^{\beta}}\right)} \right]$$

and

$$\frac{\partial \ell_2}{\partial \beta} = \sum_{i=1}^{n} (1 - \delta_i) \times$$

$$[\frac{-\frac{\alpha+1}{\alpha}\left(\lambda x_i{}^{\beta} \ln x_i \, e^{-\lambda x_i{}^{\beta}} - x_i{}^{\beta} \ln x_i \, e^{-\lambda x_i{}^{\beta}} + \lambda x_i{}^{2\beta} \ln x_i \, e^{-\lambda x_i{}^{\beta}}\right)}{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-\lambda x_i{}^{\beta}} - x_i{}^{\beta} e^{-\lambda x_i{}^{\beta}}\right) - \frac{1}{\alpha}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-\lambda x_i{}^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1) x_i{}^{\beta}}\right)\right)\right]}]$$

$$-\sum_{i=1}^{n} (1 - \delta_i) \times$$

$$[\frac{\frac{\alpha+1}{\alpha^2} \lambda x_i{}^{\beta} \ln x_i \, (e^{-\lambda x_i{}^{\beta}} - e^{-\lambda(\alpha+1) x_i{}^{\beta}})}{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-\lambda x_i{}^{\beta}} - x_i{}^{\beta} e^{-\lambda x_i{}^{\beta}}\right) - \frac{1}{\alpha}[\frac{\alpha+1}{\alpha}\left(1 - e^{-\lambda x_i{}^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1) x_i{}^{\beta}}\right)\right)]}].$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{\partial \ell_1}{\partial \lambda} + \frac{\partial \ell_2}{\partial \lambda} = 0$$

where

$$\frac{\partial \ell_1}{\partial \lambda} = \sum_{i=1}^{n} \delta_i \left[\frac{2}{\lambda} - x_i{}^{\beta} + \frac{\frac{1}{\alpha\lambda^2} - \left(\frac{1}{\alpha\lambda^2} e^{-\lambda x_i{}^{\beta}} + \frac{x_i{}^{\beta}}{\alpha\lambda} e^{-\lambda x_i{}^{\beta}} \right)}{x_i{}^{\beta} - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda x_i{}^{\beta}}\right)} \right]$$

and

$$\frac{\partial \ell_2}{\partial \lambda} = \sum_{i=1}^{n} (1 - \delta_i) \times$$

$$\left[\frac{-\frac{2(\alpha+1)}{\alpha}\left(x_i{}^{\beta} e^{-\lambda x_i{}^{\beta}}\right) - \frac{\alpha+1}{\alpha^2} x_i{}^{\beta} \left(e^{-\lambda x_i{}^{\beta}} - e^{-\lambda(\alpha+1) x_i{}^{\beta}}\right)}{1 - \left(\frac{\alpha+1}{\alpha}\right)\left(1 - e^{-\lambda x_i{}^{\beta}} - x_i{}^{\beta} e^{-\lambda x_i{}^{\beta}}\right) - \frac{1}{\alpha}\left[\frac{\alpha+1}{\alpha}\left(1 - e^{-\lambda x_i{}^{\beta}} - \frac{1}{\alpha+1}\left(1 - e^{-\lambda(\alpha+1) x_i{}^{\beta}}\right)\right)\right]}\right].$$

In the case of TWD distribution, the estimation of parameters can be obtained by numerical methods.

Bayesian inference and Confidence Interval for credibility: In Bayesian theory, given the fact that, we do not know the actual value of the parameter, by choosing an estimator, loss will be occurred. This loss can be analyzed and expressed by using a function in terms of an unknown parameter and its corresponding estimator. Four loss functions and the associated Bayesian estimators are presented below. See Calabria and Polisseni (1996).

1- Squared error loss function $L\left(\gamma(\theta), d(\underline{x})\right) = \left(d(\underline{x}) - \gamma(\theta)\right)^2$

Bayesian estimator: $d_B(\underline{x}) = E\left(\gamma(\theta)|\underline{x}\right)$

2- Absolute value loss function $L\left(\gamma(\theta), d(\underline{x})\right) = |d(\underline{x}) - \gamma(\theta)|$

Bayesian estimator:

$$d_B(\underline{x}) = Median\left(\gamma(\theta)|\underline{x}\right)$$

3- Linex loss function

$$L\left(\gamma(\theta), d(\underline{x})\right) = \left[e^{c(d(\underline{x}) - \gamma(\theta)} - c\left(d(\underline{x}) - \gamma(\theta)\right) - 1\right]$$

Bayesian estimator:

$$d_B(\underline{x}) = -\frac{1}{c} ln\left[E(e^{-c\gamma(\theta)}|\underline{x})\right]$$

4- Generalized entropy loss function

$$L\left(\gamma(\theta), d(x)\right) = \left[\left(\frac{d(x)}{\gamma(\theta)}\right)^c - c\, ln\left(\frac{d(x)}{\gamma(\theta)}\right) - 1\right]$$

Bayesian estimator:

$$d_B(\underline{X}) = \left(E\left[\gamma^{-c}(\theta)|\underline{x}\right]\right)^{-\frac{1}{c}}$$

Given that the parameters of TWD distribution are non-negative, independent prior distributions is considered for each parameter as the following

$\lambda \sim Gamma(h, g)$ ، $\alpha \sim Gamma(b, c)$ ، $\beta \sim Gamma(d, e)$

where b, c, d, e, h and g are positive.

The joint prior density function is formulated as follow:

$$\pi(\alpha, \beta, \lambda) = \frac{g^h c^b e^d}{\Gamma(h)\Gamma(b)\Gamma(c)} \lambda^{h-1} \alpha^{b-1} \beta^{c-1} e^{-(g\lambda + ac + \beta e)}.$$

Bayesian inference for complete data set:

Suppose $X_1, \ldots, X_n$ be a random sample from $TWD(\alpha, \beta, \lambda)$, then the posterior distribution is given as

$$\pi^*(\alpha, \beta, \lambda|\underline{x}) \propto \pi(\alpha, \beta, \lambda) f\left(\underline{x}, \alpha, \beta, \lambda\right)$$

then, it results that

$$\pi^*\left(\alpha, \beta, \lambda|\underline{x}\right) = \frac{\lambda^{h-1} a^{b-1} \beta^{c-1} e^{-(g\lambda + ac + \beta e)} \prod_{i=1}^n \left(\frac{\alpha+1}{\alpha} \lambda^2 \beta x_i^{\beta-1} e^{-\lambda x_i^\beta} \left(x_i^\beta - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha x_i^\beta}\right)\right)\right)^{\delta_i}}{D}$$

$$\times \frac{\prod_{i=1}^n (x_i^\beta - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha x_i^\beta}\right))}{D}.$$

where D is determined from the following formula:

$$D = \int_0^\infty \int_0^\infty \int_0^\infty \lambda^{h-1} a^{b-1} \beta^{c-1} e^{-(g\lambda + ac + \beta e)} \prod_{i=1}^n \frac{\alpha+1}{\alpha} \lambda^2 \beta x_i^{\beta-1} e^{-\lambda x_i^\beta} \left(x_i^\beta - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha x_i^\beta}\right)\right) da\, d\beta\, d\lambda.$$

The Bayesian estimators for each parameter of the TWD distribution under the above-mentioned loss function are not explicit. Consequently, simulation of the posterior distributions is feasible through using the MCMC algorithms such as Gibbs sampling method and Metropolis-Hastings algorithm and then the associated Bayesian estimators and Bayesian credible confidence interval are calculated. The Bayesian estimators for the parameters α, β, λ are represented by $\widehat{\alpha}_B$, $\hat{\beta}_B$, $\hat{\lambda}_B$, respectively

4.2.2. Bayesian inference for right censored data

Suppose $(X_1, \delta_1), \ldots, (X_n, \delta_n)$ be a right censored random sample from $TWD(\alpha, \beta, \lambda)$, then the posterior distribution is given as

$$\pi^*(\alpha, \beta, \lambda|\underline{x}, \underline{\delta}) \propto \pi(\alpha, \beta, \lambda) f\left(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda\right)$$

then, it results that

$$\pi^*\left(\alpha, \beta, \lambda|\underline{x}, \underline{\delta}\right) = \frac{\lambda^{h-1} a^{b-1} \beta^{c-1} e^{-(g\lambda + ac + \beta e)} \prod_{i=1}^n \frac{\alpha+1}{\alpha} \lambda^2 \beta x_i^{\beta-1} e^{-\lambda x_i^\beta}}{D}$$

$$\times \frac{\prod_{i=1}^n \left(1 - wF_1(x_i) + (1-w)F_2(x_i)\right)^{1-\delta_i}}{D}.$$

where D is determined from the following formula:

$$D = \int_0^\infty \int_0^\infty \int_0^\infty [\,\lambda^{h-1}\, a^{b-1}\, \beta^{c-1}\, e^{-(g\lambda + ac + \beta e)} \times \prod_{i=1}^{n} \left(\frac{\alpha+1}{\alpha} \lambda^2 \beta {x_i}^{\beta-1} e^{-\lambda {x_i}^{\beta}} \left({x_i}^{\beta} - \frac{1}{\alpha\lambda}\left(1 - e^{-\lambda\alpha {x_i}^{\beta}}\right)\right)\right)^{\delta_i} \times \prod_{i=1}^{n} \left(1 - wF_1(x_i) + (1-w)F_2(x_i)\right)^{1-\delta_i}]\,da\, d\beta\, d\lambda$$

Analogous previous subsection, we use the MCMC algorithms such as Gibbs sampling method and Metropolis-Hastings algorithm and then the associated Bayesian estimators and Bayesian credible confidence interval are calculated.

Results

In this section, we illustrate the usefulness of the TWD distribution. We fit proposed distribution to real data sets in complete and censored cases by ML method and compare the results with Weibull and generalized exponential (GE) with respective densities

$$f_{Weibull}(x) = \frac{\beta}{\lambda^\beta} x^{\beta-1} e^{-(\frac{x}{\lambda})^\beta}, \quad x \geq 0,$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}, \quad x \geq 0\,,$$

Furthermore, in this section, we provide Bayesian estimation analysis of parameters of TWD for two real data sets.

Censored data set: Meeker and Escobar (2014) represented observed lifetimes of 30 devices that includes eight censored observations.

2 10 13 23 23 28 30 65 80 88 106 143 147 173 181 212 245 247 261 266 275 293 300+ 300+ 300+ 300+ 300+ 300+ 300+ 300+.

The + sign indicates right-ensured observations.

Complete data set: The data set has been obtained from Bader and Priest (1982), and it representsthe strength for the single carbon fibers and impregnated 1000-carbon fiber tows, measured in GPa. We report the data of single carbon fiber tested at gauge length 1mm. The data are presented below:

2.247 2.64 2.908 3.099 3.126 3.245 3.328 3.355 3.383 3.572 3.581 3.681 3.726 3.727 3.728 3.783 3.785 3.786 3.896 3.912 3.964 4.05 4.063 4.082 4.111 4.118 4.141 4.246 4.251 4.262 4.326 4.402 4.457 4.466 4.519 4.542 4.555 4.614 4.632 4.634 4.636 4.678 4.698 4.738 4.832 4.924 5.043 5.099 5.134 5.359 5.473 5.571 5.684 5.721 5.998 6.06

Before analyzing this data set, we use the scaled-TTT plot to verifiy our model validity, see Aarset (1987). It allows to identify the shape of hazard function graphically. We provide the empirical scaled-TTT plot of above data set. Fig. 5. Shows the scaled-TTT plot is concave. It indicates that the hazard function is increasing; therefore it verifies our model validity.

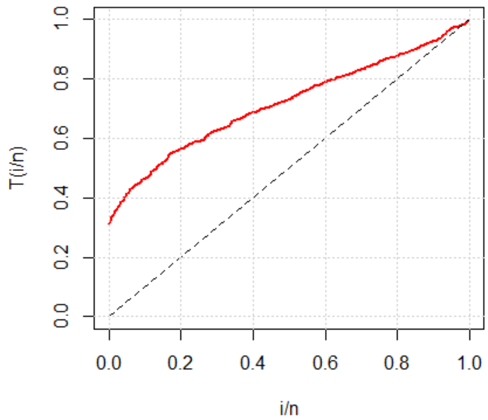

Figure 5. Scaled-TTT plot of the strength for the single carbon fibers.

Analysis results for censored data set

Here, we fit the TWD distribution to the censored data set. Table 1 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC) and Bayesian Akaike information criterion (BIC) for censored data set.

Analysis results for complete data set

Table 1. The MLEs of parameters for lifetimes of devices data.

Model	MLEs of parameters	Log-likelihood	AIC	BIC
TWD	$\hat{\alpha}$ =10.872, $\hat{\beta}$ = 0.5203, $\hat{\lambda}$ =0.1281673	-143.096	292.1921	296.3957

Table 2. The MLEs of parameters for Guinea pigs' data

Model	MLEs of parameters	Log-likelihood	AIC	KS (P-value)
TWD	$\hat{\alpha}$ =0.276, $\hat{\beta}$ =3.138, $\hat{\lambda}$ =0.026	-67.91	141.82	0.067 (0.951)
Weibull	$\hat{\beta}$ =5.705, $\hat{\lambda}$ = 29.514	-68.93	141.91	0.090 (0.717)
GE	$\hat{\alpha}$ = 1.325, $\hat{\lambda}$ =0.043	-71.77	147.56	0.095 (0.655)

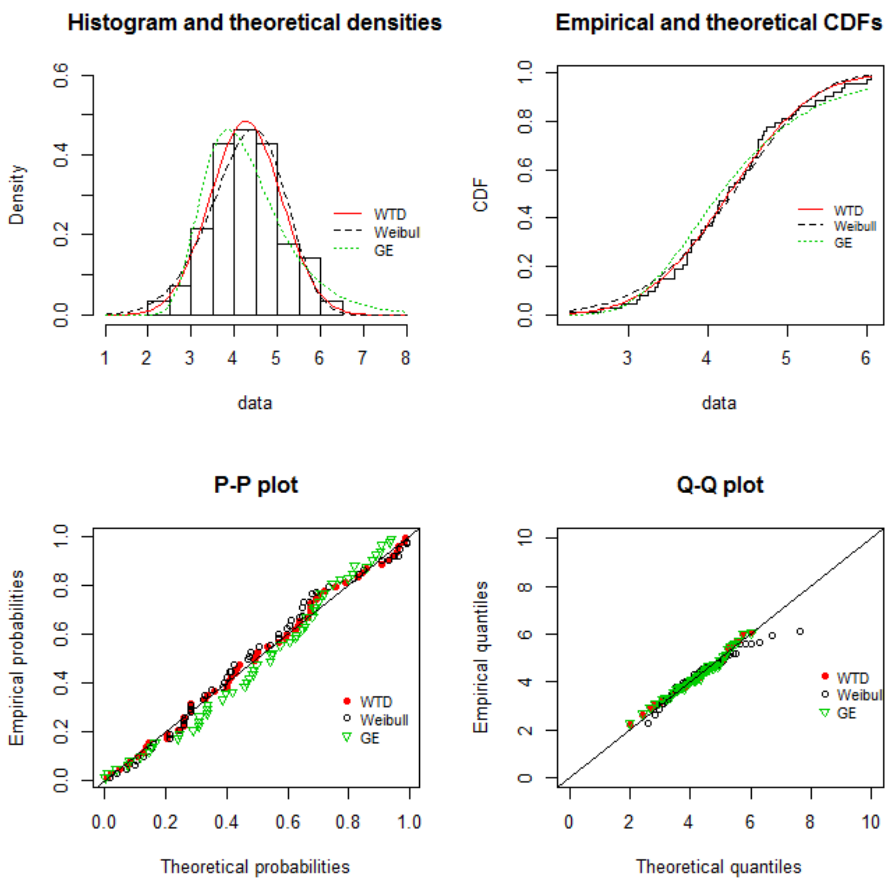

Figure 6. Histogram and fitted density plots, the plots of empirical and fitted $cdfs$, P-P plots and Q-Q plots for the real strength of the single carbon fibers data.

Here, we fit the TWD distribution to the complete data set and compare it with the generalized exponential and Weibull densities. Table 2 includes the MLE's of parameters, Kolmogorov-Smirnov (K-S) distance between the empirical distribution and the fitted model, its corresponding p-value, log-likelihood and Akaike information criterion (AIC) for the real data set. The selection criterion is that the lowest AIC and $K-S$ statistic corresponding to the best fitted model. The TWD distribution provides the best fit for the data set as it has lower AIC and K-S statistic than the other competitor models. The histogram of data set, fitted pdf of the TWD distribution and fitted pdfs of other competitor distributions for the real data set are plotted in Figure 6. Also, the plots of empirical and fitted cdfs functions, P-P plots and Q-Q plots for the TWD and other fitted distributions are displayed in Figure 6. These plots also support the results in Table 2.

Bayesian analysis results

In this section, the numerical analyzes of Bayesian estimators are presented for the data sets that described in the beginning of section 5. These estimators are obtained for both complete and censored data under the four loss functions that considered in subsection 4.2.

Table 3. Bayesian estimation of parameters for complete data set

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
Mle and confidence interval	0.276	3.138	0.026
	(0 ,10)	(2.196,4.07)	(0 , 0.1204)
	Bayesian estimation under the squared loss function		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	0.2046957	3.551725	0.01104583
	Bayesian estimation under the absolute value loss function		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	0.2033	3.538	0.009
	Bayesian estimation under the Linex loss function for $c = 3$		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	0.2046656	3.421548	0.0110
Bayesian estimation and HPD and credible intervals	Bayesian estimation under the Generalized entropy loss function for $c = 3$		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	0.007	0.2045086	3.499393
		HPD CI	
	(0.2013, 0.207)	(3.324, 3.778)	(0.007, 0.0142)
		Crediable CI	
	(0.2, 0.2139)	(3.078, 4.235)	(0.0215, 0.003)

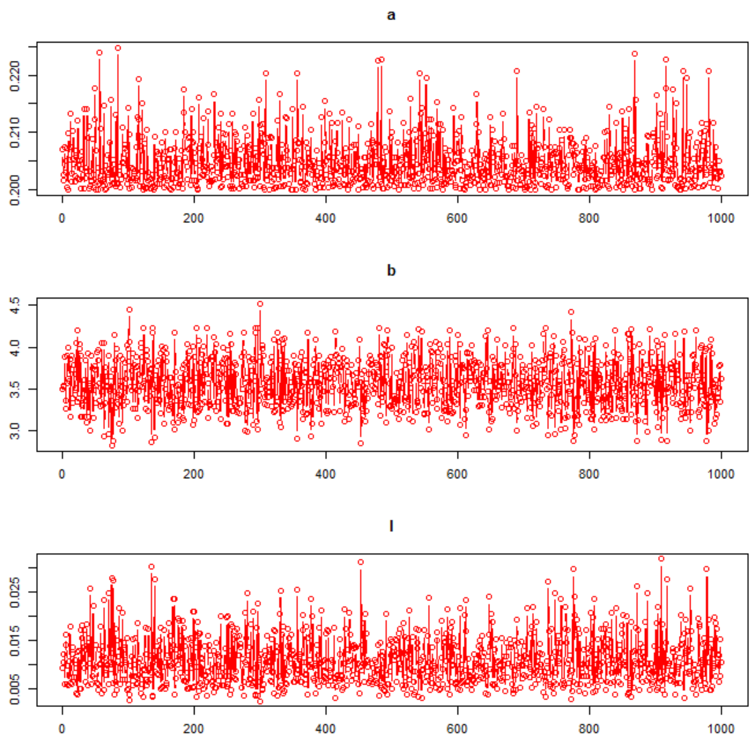

Figure 7. Plots of history of posterior samples of each parameter of TWD distribution for complete data set.

Bayesian analysis results for complete and right censored data set

Table 3 is devoted to the numerical results of the Bayesian estimations for the complete data set. This table shows the Bayesian estimator and 95% credible and highest posterior density (HPD) intervals for each parameter of proposed new model TWD. In addition, the maximum likelihood estimator and corresponding asymptotic confidence intervals are calculated in order to compare with corresponding Bayesian intervals. Plots of history (Trace plot) of posterior samples, autocorrelation function (acf) plots of posterior samples and histogram of

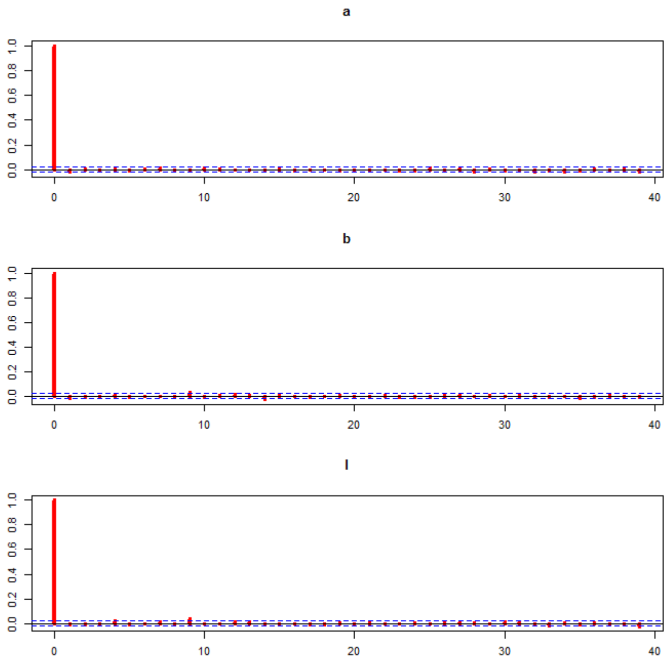

Figure 8. ACF plots of posterior samples of each parameter of TWD distribution for complete data set.

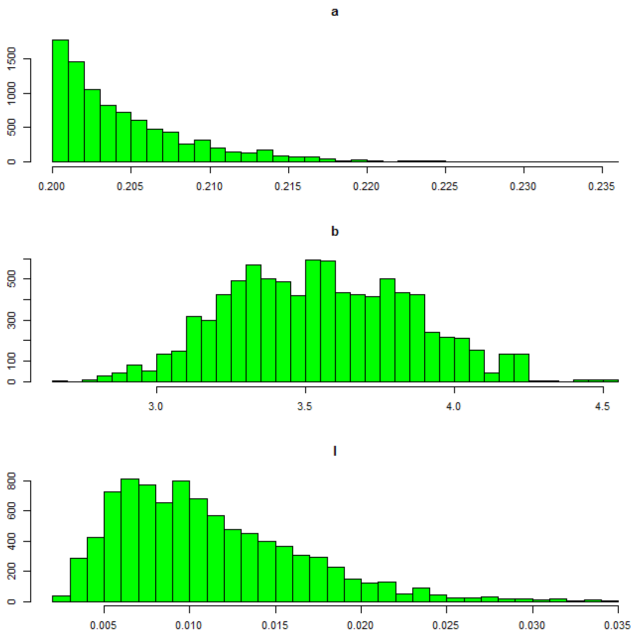

Figure 9. Histogram of posterior samples of each parameterof TWD distribution for complete data set.

posterior samples of each parameter of TWD distribution provided in Figure 7, 8 and 9 respectively. These figures show that the simulation processes of Gibbs algorithm has been of good quality.

Bayesian analysis results right censored data set

The results of this section are similar to the previous one, with the difference that the Bayesian analysis for censored data is carried out. Table 4 is devoted to the numerical results of the Bayesian estimations for the censored data set. This table shows the Bayesian estimator and

Table 4. Bayesian estimation of parameters for the censored data set

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
Mle and confidence interval	10.8718620	0.5203	0.1281673
	(0 , 64.26)	(0.283, 0.757369)	(0 , 0.3094049)
Bayesian estimation and HPD and credible intervals	Bayesian estimation under the squared loss function		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	10.38368	0.5612756	0.2566403
	Bayesian estimation under the absolute value loss function		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	10.32	0.5559	0.2458
	Bayesian estimation under the Linex loss function for $c = 3$		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	9.667699	0.5536759	0.2435577
	Bayesian estimation under the Generalized entropy loss function for $c = 3$		
	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
	10.24125	0.5430148	0.1830114
	HPD CI		
	(9.62675, 11.12000)	(0.5120, 0.6071)	(0.186300, 0.315225)
	Crediable CI		
	(9.008, 11.820)	(0.4277, 0.7088)	(0.08629, 0.44410)

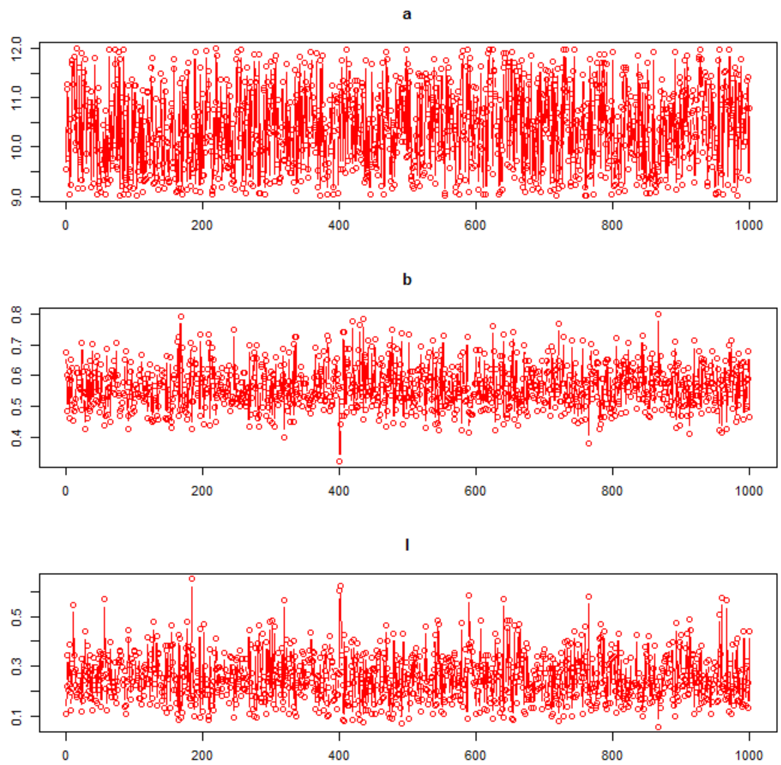

Figure 10. Plots of history of posterior samples of each parameter of TWD distribution for censored data set.

95% credible and HPD intervals provided for each parameter of proposed TWD model.

In addition, the maximum likelihood estimator and corresponding asymptotic confidence intervals are calculated in order to compare with corresponding Bayesian intervals. Plots of history of posterior samples, acf plots of posterior samples and histogram of posterior samples of each parameter of TWD distribution provided in Figures 10, 11 and 12 respectively. These figures show that the simulation processes of Gibbs algorithm has been of good quality.

Conclusion

In this paper, we have proposed a new transformed Weibull distribution, denoted by TWD. It is investigated that the new model has

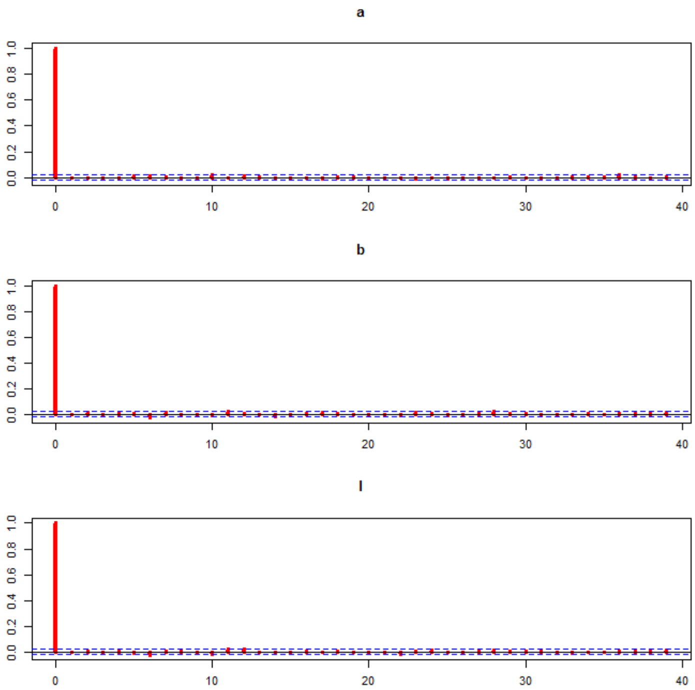

Figure 11. ACF plots of posterior samples of each parameter of TWD distribution for censored data set.

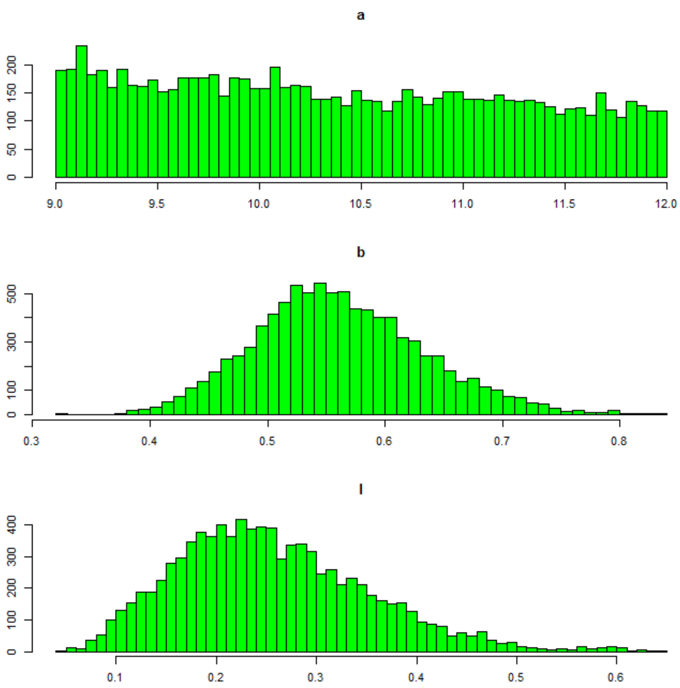

Figure 12. Histogram of posterior samples of each parameter of TWD distribution for censored data set.

increasing, decreasing and bathtub shape hazard functions. We provide the comprehensive Bayesian and maximum likelihood estimation procedures for complete and right censored real observations.

Graphically, the plots of Gibbs algorithm are provided for sampling of the posterior distributions of parameters, and these plots confirm the numerical results that given in tables.

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Conflict of Interests

Authors have no conflict of interests.

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