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Original Article

A Two-Parameter Pranav Distribution with Properties and Its Application.

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ABSTRACT

Introduction: Lifetime distribution has drawn so much attention in recent research, and this has lead to the development of new lifetime distribution. Addition of parameters to the existing distribution makes the distribution more flexible and reliable and applicable model has become the focus of the recent search. This paper proposed a two-parameter Pranav distribution which has its base from a one-parameter Pranav and Ishita distribution.

Methods Two parameter Pranav distribution was proposed. Mathematical and statistical properties of the distribution which includes; moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviation, Bonferroni and Lorenz curves were developed. Other lifetime distributions such as Ishita, Akash, Sujatha, Shanker, Lindley, and Exponential distributions were considered in the study.

Results: This new distribution was compared with two-parameter Akash, Lindley, one parameter Pranav, Ishita, Akash, Sujatha, Shanker, Lindley, and Exponential distributions to determine the efficiency of the new model. The estimation of parameters has been X-rayed using the method of moments and maximum likelihood. Also, AIC and BIC were used to test for the goodness of fit of the model which was applied to a real-life data of hypertensive patients. The results show that the new two-parameter Pranav distribution has the lowest value of AIC and BIC

Conclusion: Based on the AIC and BIC values we can conclude that the two-parameter Pranav is more efficient than the other distribution for modeling survival of hypertensive patients. Hence two-parameter Pranav can be seen as an important distribution in modeling lifetime data.

Introduction

Application of lifetime data for modeling and predicting the survival of individual has become so popular in the field of medicine, engineering, management science and most especially biological science. The popularity of this lifetime data has to lead to development of new lifetime distribution and performances of the distribution to lifetime data seems better than the existing ones such as exponential, gamma, lognormal and so on. Development of one parameter lifetime distribution has always be popular such as Lindley, Akash, Ishita and the rest but the addition of new parameter to the distributions will make the distribution better than the existing ones. As there is always a problem of fitness to the data, adding a new parameter will reduce such problem and the distribution will be more flexible. The aim of this paper is to derive a

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continuous distribution that would be flexible, robust and reliable than the Pranav distribution.

A one-parameter distribution, known as Pranav distribution, given by its probability density function

$$f(x;\theta) = \frac{\theta^4}{\theta^4+6}(\theta + x^3)e^{-\theta x}; x > 0, \theta > 0 \quad 1.1$$

Was introduced by Shukla (2018) The probability distribution function (pdf) is a mixture of two distributions, exponential distribution having scale parameter θ and gamma distribution having shape parameter 4 and scale parameter θ[1]. Its cumulative distribution function has been obtained as

$$F(x;\theta) = 1 - \left[1 + \frac{\theta x(\theta^2x^2+3\theta x+6)}{\theta^4+6}\right]e^{-\theta x}; x > 0, \theta > 0 \quad 1.2$$

Major properties of this distribution were generated and showed that (1.1) provides a better model than the Ishita(2017), Sujatha(2016), Akash(2015), Shanker(2015), Lindley(1958) and exponential distribution as regard to the application. The first four moments about the origin of the Pranav distribution was gotten as

$$\mu_1' = \frac{\theta^4 + 24}{\theta(\theta^4 + 6)} \quad \mu_2' = \frac{2(\theta^4 + 60)}{\theta^2(\theta^4 + 6)} \quad \mu_3' = \frac{6(\theta^4 + 120)}{\theta^3(\theta^4 + 6)} \quad \mu_4' = \frac{24(\theta^4 + 210)}{\theta^4(\theta^4 + 6)}$$

and its central moments have been obtained as

$$\mu_2 = \frac{(\theta^8 + 84\theta^4 + 144)}{\theta^2(\theta^4 + 6)^2} \quad \mu_3 = \frac{2(\theta^{12} + 198\theta^8 + 324\theta^4 + 864)}{\theta^3(\theta^4 + 6)^3}$$

$$\mu_4 = \frac{9(\theta^{16} + 312\theta^{12} + 2304\theta^8 + 10368\theta^4 + 10368)}{\theta^4(\theta^4 + 6)^4}$$

He went ahead and generated the statistical properties including shapes of density function for varying values of parameters, hazard rate function, mean residual life function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves and stress-strength reliability. The application of Pranav distribution was showed, by comparing it with other distributions after he discussed the maximum likelihood estimation of the parameter

Another distribution was also proposed which was named as Ishita distribution having parameter θ and its probability density function (pdf) is given by

$$f(x;\theta) = \frac{\theta^3}{\theta^3+2}(\theta + x^2)e^{-\theta x};\ x > 0, \theta > 0 \qquad 1.3$$

Ishita distribution is a combination of exponential (θ) and a gamma (3,θ) distribution with mixing proportion $\frac{\theta^3}{\theta^3+2}$.

The statistical properties including shapes of pdf, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual function, stress- strength reliability, mean deviation, stochastic ordering and Bonferroni and Lorenz curves and some others. The estimation of a parameter using maximum likelihood and method of moments has been discussed as well and its application to real-life data was also showed and they also established its superiority over exponential, Lindley and Akash distributions[2]

In this paper, a two-parameter Pranav distribution, of which the Ishita and Pranav distribution is a special case, has been suggested. Its first four moments and some of the related measures have been obtained. Its failure rate, mean residual rate function, stochastic ordering, mean deviations, Bonferroni and Lorenz curves have been discussed. The estimation of the parameters has been discussed using the method of moments and the method of maximum likelihood. This new distribution was developed in other to have a more flexible distribution that will be a good fit to the data because the higher the parameter the more flexible the distribution will be.

Numerical examples have been presented to test the goodness of fit of the two-parameter Pranav distribution over exponential, one parameter Pranav, Lindley, Sujatha, Ishita, and Akash distributions.

Materials and Methods

A two-parameter Pranav distribution having parameter θ and α can be defined by its probability density function (pdf)

$$f(x;\theta,\alpha) = \frac{\theta^4}{\alpha\theta^4+6}(\alpha\theta + x^3)e^{-\theta x};\ x > 0, \theta > 0, \alpha \geq 0. \qquad 2.1$$

where θ is a scale parameter and α is a shape parameter. It reduces to Pranav distribution (1.1) for α =1. The pdf (2.1) can be shown as a combination of exponential (θ) and gamma (4, θ) distributions as follows:

$$f(x;\theta,\alpha) = Qf_1(x) + (1-Q)f_2(x) \qquad 2.2$$

where

$$Q = \frac{\alpha\theta^4}{\alpha\theta^4+6}, \quad f_1(x) = \theta e^{-\theta x} and f_2(x) = \frac{\theta^4 x^3 e^{-\theta x}}{6}$$

$$f(x;\theta,\alpha) = \frac{\alpha\theta^4}{\alpha\theta^4+6}\theta e^{-\theta x} + (1 - \frac{\alpha\theta^4}{\alpha\theta^4+6})\frac{\theta^4 x^3 e^{-\theta x}}{6} \qquad 2.3$$

$$= \frac{\alpha\theta^4}{\alpha\theta^4+6}\theta e^{-\theta x} + \frac{\alpha\theta^4+6-\alpha\theta^4}{\alpha\theta^4+6} . \frac{\theta^4 x^3 e^{-\theta x}}{6} = \frac{\alpha\theta^4}{\alpha\theta^4+6}\theta e^{-\theta x} + \frac{\theta^4 x^3 e^{-\theta x}}{\alpha\theta^4+6}$$

$$f(x;\theta,\alpha) = \frac{\theta^4}{\alpha\theta^4+6}(\alpha\theta + x^3)e^{-\theta x};\ x > 0, \theta > 0, \alpha \geq 0. \qquad 2.4$$

To confirm that it is a proper pdf

Recall that $\int_{-\infty}^{\infty} f(x)dx = 1$

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$$\int_0^\infty \frac{\theta^4}{\alpha\theta^4 + 6}(\alpha\theta + x^3)e^{-\theta x}dx$$

$$\frac{\theta^4}{\alpha\theta^4 + 6}\int_0^\infty (\alpha\theta + x^3)e^{-\theta x}dx$$

Using integral by parts and putting the necessary limits gives the above as

$$\frac{\theta^4}{\alpha\theta^4 + 6}\left[-\alpha e^{-\theta x}|_0^\infty + \frac{6}{\theta^3}\left(-\frac{1}{\theta}e^{-\theta x}\right)|_0^\infty\right]$$

$$\frac{\theta^4}{\alpha\theta^4 + 6} * \frac{\alpha\theta^4 + 6}{\theta^4} = 1$$

Derivation of the cumulative distribution function (CDF) is given as

$$\int_0^x f(t)dt = F(x) \qquad 2.5$$

$$F(x) = \int_0^x \frac{\theta^4}{\alpha\theta^4+6}(\alpha\theta + t^3)e^{-\theta t}dt \ = \frac{\theta^4}{\alpha\theta^4+6}\int_0^x (\alpha\theta + t^3)e^{-\theta t}dt \qquad 2.6$$

Using integral by parts and putting the necessary limits gives the above as

$$= -1 - \left[1 + \frac{\theta x(\theta^2 x^2 + 3\theta x + 6)}{\alpha\theta^4 + 6}\right] e^{-\theta x}$$

Multiplying by -1 gives below

$$F(x) = 1 - \left[1 + \frac{\theta x(\theta^2 x^2 + 3\theta x + 6)}{\alpha\theta^4 + 6}\right] e^{-\theta x} \qquad 2.7$$

The corresponding cumulative distribution function (CDF) of (2.1) can be expressed as

$$F(x) = 1 - \left[1 + \frac{\theta x(\theta^2 x^2 + 3\theta x + 6)}{\alpha\theta^4 + 6}\right] e^{-\theta x} \;; x > 0, \theta > 0, \alpha \geq 0 \qquad 2.8$$

The Pdf and Cdf behavior of the two-parameter Pranav distribution concerning various combinations of parameters θ and α

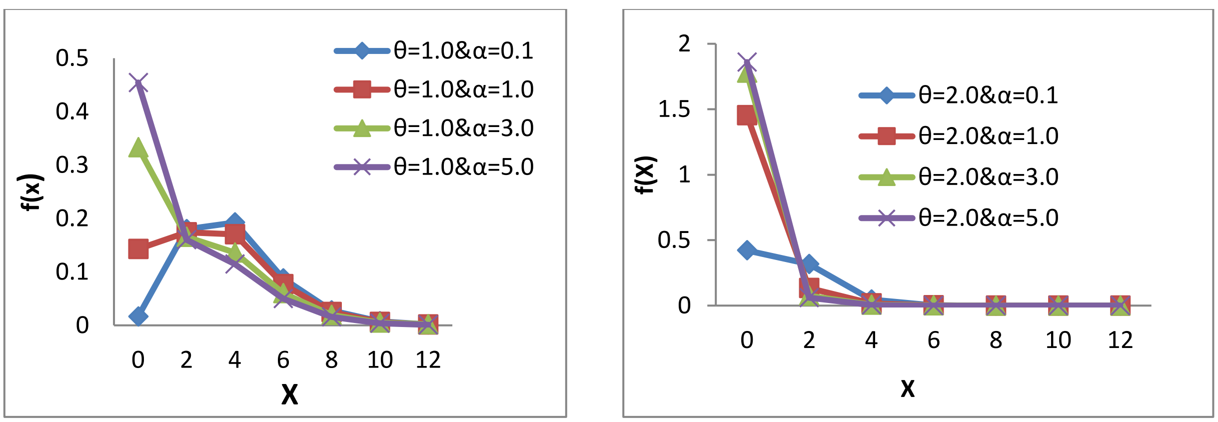


Figure1. *Graphs of the pdf of the two-parameter Pranav distribution with different values of parameters.*

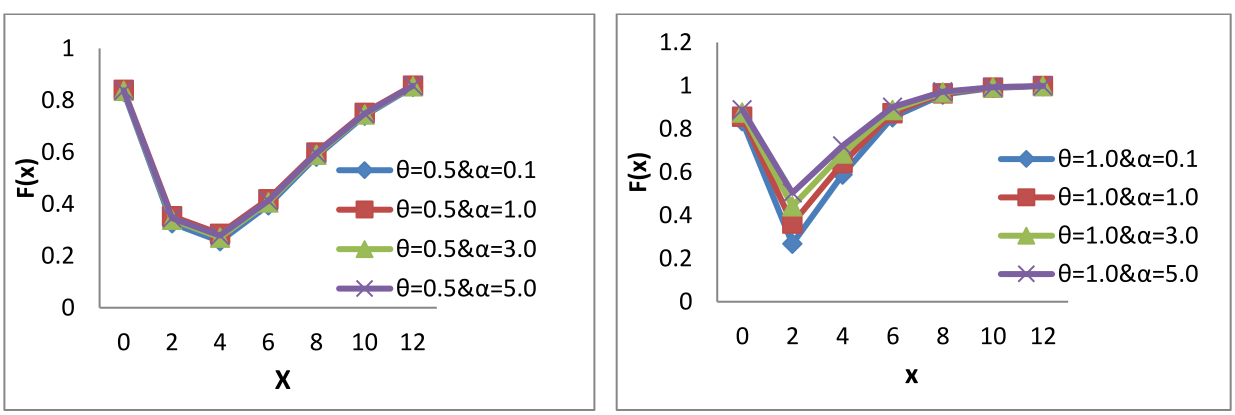


Figure2. *Graphs of the CDF of the two-parameter Pranav distribution with different values of parameters.*

Moments and Related Measures

The rth moment about the origin of the distribution is gotten by obtaining the Moment generating function first.

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \qquad 2.9$$

$$= \int_0^\infty e^{tx} \frac{\theta^4}{\alpha\theta^4+6}(\alpha\theta + x^3)e^{-\theta x}dx \qquad 2.10$$

$$= \frac{\theta^4}{\alpha\theta^4+6}\int_0^\infty (\alpha\theta + x^3)e^{-x(\theta-t)}dx$$

$$= \frac{\theta^4}{\alpha\theta^4+6}\left[\frac{\alpha\theta}{\theta-t} + \frac{6}{(\theta-t)^4}\right]$$

$$= \frac{\theta^4}{\alpha\theta^4+6}\left[\frac{\alpha\theta}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k + \frac{6}{\theta^4}\sum_{k=0}^{\infty}\binom{k+3}{k}\left(\frac{t}{\theta}\right)^k\right]$$

$$M_x(t) = \sum_{k=0}^{\infty} \frac{\alpha\theta^4+(k+1)(k+2)(k+3)}{\alpha\theta^4+6}\left(\frac{t}{\theta}\right)^k \qquad 2.11$$

The rth moment about the origin of the two-parameter Pranav distribution is obtained as

$$\boldsymbol{\mu_r'} = \frac{r!\left[\alpha\theta^4+(r+1)(r+2)(r+3)\right]}{\theta^r(\alpha\theta^4+6)} \;;\boldsymbol{r = 1, 2, 3} \ldots .... \qquad 2.12$$

Thus, the first four moments about the origin of the two-parameter Pranav distribution are given as

$$\boldsymbol{\mu_1'} = \frac{\alpha\theta^4+24}{\theta(\alpha\theta^4+6)} \qquad \boldsymbol{\mu_2'} = \frac{2(\alpha\theta^4+60)}{\theta^2(\alpha\theta^4+6)} \qquad 2.13$$

$$\boldsymbol{\mu_3'} = \frac{6(\alpha\theta^4+120)}{\theta^3(\alpha\theta^4+6)} \qquad \boldsymbol{\mu_4'} = \frac{24(\alpha\theta^4+210)}{\theta^4(\alpha\theta^4+6)} \qquad 2.14$$

Applying the relationship between moments about the mean and moments about origin gives the moments about the mean of two-parameter Pranav distribution as

$$\mu_2 = E[(X-\mu)^2] = \mu_2' - \mu^2$$

$$\mu_2 = \frac{(\alpha^2\theta^8+84\alpha\theta^4+144)}{\theta^2(\alpha\theta^4+6)^2} \qquad 2.15$$

$$\mu_3 = E[(X-\mu)^3] = \mu_3' - 3\mu\mu_2' + 2\mu^3$$

$$\mu_3 = \frac{2(\alpha^3\theta^{12}+198\alpha^2\theta^8+324\alpha\theta^4+864)}{\theta^3(\alpha\theta^4+6)^3} \qquad 2.16$$

$$\mu_4 = E[(X-\mu)^4] = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$

$$\mu_4 = \frac{9(\alpha^4\theta^{16}+312\alpha^3\theta^{12}+2304\alpha^2\theta^8+10368\alpha\theta^4+10368)}{\theta^4(\alpha\theta^4+6)^4} \qquad 2.17$$

The coefficient of variation (C.V), coefficient of skewness ($\sqrt{\boldsymbol{\beta_1}}$), coefficient of kurtosis ($\boldsymbol{\beta_2}$), index of dispersion ($\boldsymbol{\gamma}$) of the two-parameter Pranav distribution are given as $c.v = \frac{\sigma}{\mu_1'}$ where σ is the standard deviation which is the square root of variance.

$$\sigma^2 = \mu_2 = \frac{(\alpha^2\theta^8+84\alpha\theta^4+144)}{\theta^2(\alpha\theta^4+6)^2} \text{ and } \mu_1' = \frac{\alpha\theta^4+24}{\theta(\alpha\theta^4+6)}$$

$$\boldsymbol{c.v = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\alpha^2\theta^8+84\alpha\theta^4+144}}{\alpha\theta^4+24}} \qquad 2.18$$

$$\sqrt{\boldsymbol{\beta_1}} = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{2(\alpha^3\theta^{12}+198\alpha^2\theta^8+324\alpha\theta^4+864)}{(\alpha^2\theta^8+84\alpha\theta^4+144)^{3/2}} \qquad 2.19$$

$$\boldsymbol{\beta_2 = \frac{\mu_4}{\mu_2} = \frac{9(\alpha^4\theta^{16}+312\alpha^3\theta^{12}+2304\alpha^2\theta^8+10368\alpha\theta^4+10368)}{(\alpha^2\theta^8+84\alpha\theta^4+144)^2}} \qquad 2.20$$

$$\boldsymbol{\gamma = \frac{\sigma^2}{\mu_1} = \frac{\alpha^2\theta^8+84\alpha\theta^4+144}{\theta(\alpha\theta^4+6)(\alpha\theta^4+24)}} \qquad 2.21$$

It can be easily verified that these statistical constants of the two-parameter Pranav distribution reduce to the corresponding statistical constants of Pranav distribution at $\boldsymbol{\alpha = 1}$.

2.2 Failure rate and Mean residual life function

Let X be a continuous random variable with pdf $\boldsymbol{f}(\boldsymbol{x})$ and Cdf $\boldsymbol{F}(\boldsymbol{x})$. The hazard rate function (also known as failure rate function), $\boldsymbol{h}(\boldsymbol{x})$ and the mean residual function, $\boldsymbol{m}(\boldsymbol{x})$ of x are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{p(X<x+\Delta x/X>x)}{\Delta x} = \frac{f(x)}{1-F(x)} \qquad 2.22$$

And $$m(x) = E[X - x/X > x] = \frac{1}{1-F(x)} \int_x^\infty [1 - F(t)]dt \qquad 2.23$$

The corresponding failure rate function $h(x)$ and the mean residual life function $m(x)$ of the distribution are thus given as

$$h(x) = \frac{\theta^4(\alpha\theta+x^3)}{(\theta^3x^3+3\theta^2x^2+6\theta x+\alpha\theta^4+6)} \qquad 2.24$$

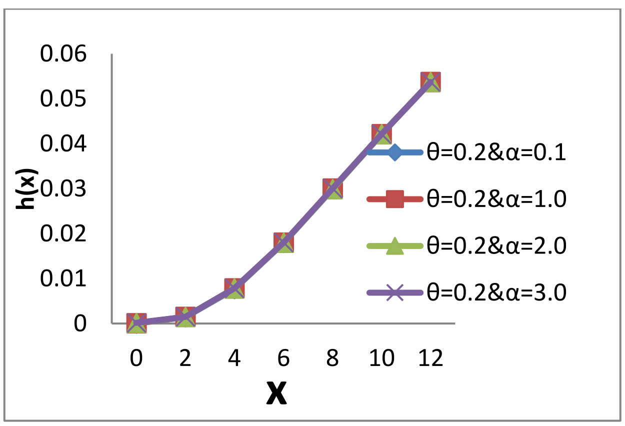
$$m(x) = \frac{1}{(\theta^3x^3+3\theta^2x^2+6\theta x+\alpha\theta^4+6)e^{-\theta x}} \int_x^\infty (\theta^3x^3 + 3\theta^2x^2 + 6\theta x + \alpha\theta^4 + 6)e^{-\theta x}$$

$$= \frac{\theta^3x^3+6\theta^2x^2+18\theta x+\alpha\theta^4+24}{\theta(\theta^3x^3+3\theta^2x^2+6\theta x+\alpha\theta^4+6)} \qquad 2.25$$

It can be verified that and $h(0) = \frac{\alpha\theta^5}{\alpha\theta^4+6} = f(0) and\, m(0) = \frac{\alpha\theta^4+24}{\theta(\alpha\theta^4+6)} = \mu_1$

The expression for $h(x)$ and $m(x)$ of the two-parameter Pranav distribution reduces to the corresponding $h(x)$ and $m(x)$ of Pranav distribution at $\alpha = 1$.

The h(x) and m(x) behavior of the two-parameter Pranav distribution concerning various combinations of parameters θ and α



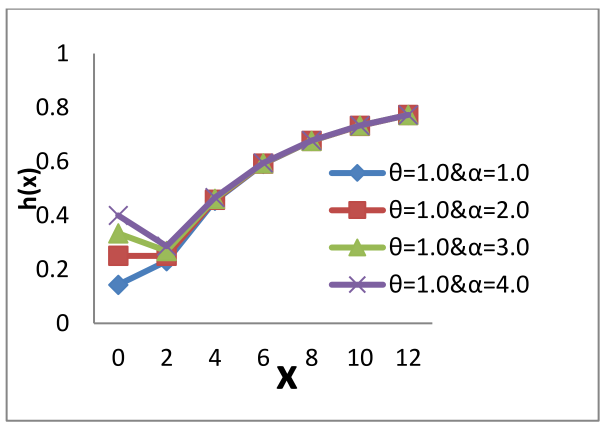


Figure3. *Graphs of the h(x) of the two-parameter Pranav distribution with different values of parameters.*

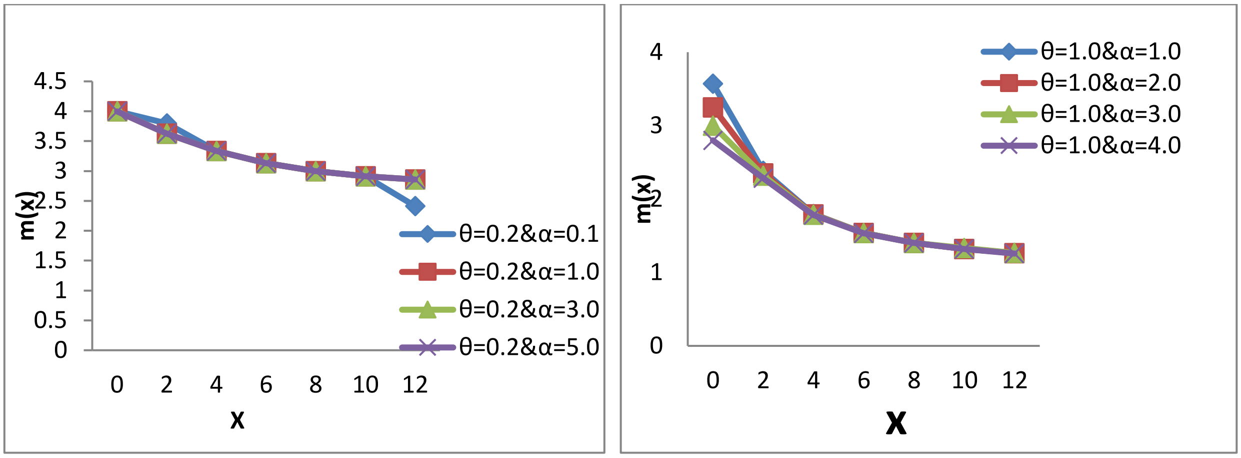


Figure4. *Graphs of the m(x) of the two-parameter Pranav distribution with different values of parameters.*

Stochastic Orderings

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. A random variable X is said to be smaller than a random variable Y in the

i.Stochastic order $\boldsymbol{X \leq_{st} Y}$ if $\boldsymbol{f_X(x) \geq f_Y(x)}$ for all x.

ii.Hazard rate order X≤_hr Y if h_X (x)≥h_Y (x) for all x.

iii.Mean residual life order $\boldsymbol{X \leq_{mrl} Y}$ if $\boldsymbol{m_X(x) \geq m_Y(x)}$ for all x.

iV.Likelihood ratio order $\boldsymbol{X \leq_{lr} Y}$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions.

$$\boldsymbol{X \leq_{lr} Y \Rightarrow X \leq_{hr} \Rightarrow X \leq_{mrl} Y}$$
$$\Downarrow$$
$$\boldsymbol{X \leq_{st} Y}$$

The two-parameter Pranav distribution (PD) is ordered with respect to the strongest likelihood ratio ordering as shown in the following theorem;

Theorem: let $\boldsymbol{X}\sim$ two-parameter PD$(\boldsymbol{\alpha_1, \theta_1})$ and $\boldsymbol{Y}\sim$ two-parameter PD $(\boldsymbol{\alpha_2, \theta_2})$. If $\boldsymbol{\alpha_1 = \alpha_2}$and $\boldsymbol{\theta_1 \geq \theta_2}$ $(\boldsymbol{or if \theta_1 = \theta_2 and \alpha_1 \geq \alpha_2})$, then $\boldsymbol{X \leq_{lr} Y}$ and hence $\boldsymbol{X \leq_{hr} Y}$, $\boldsymbol{X \leq_{mrl} Y}$ and $\boldsymbol{X \leq_{st} Y}$.

Proof: We have

$$\frac{\boldsymbol{f_X(x)}}{\boldsymbol{f_Y(x)}} = \frac{\boldsymbol{\theta_1^4(\alpha_2\theta_2 + 6)}}{\boldsymbol{\theta_2^4(\alpha_1\theta_1 + 6)}} \frac{\boldsymbol{(\alpha_1\theta_1 + x^3)}}{\boldsymbol{(\alpha_2\theta_2 + x^3)}} \boldsymbol{e^{-(\theta_1-\theta_2)x}}; \boldsymbol{x > 0}$$

now

$$\ln \frac{f_X(x)}{f_Y(x)} = \ln\left[\frac{\theta_1^4(\alpha_2\theta_2+6)}{\theta_2^4(\alpha_1\theta_1+6)}\right] + \ln\left[\frac{(\alpha_1\theta_1+x^3)}{(\alpha_2\theta_2+x^3)}\right] - (\theta_1-\theta_2)x$$

$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{3x^2(\alpha_2\theta_2-\alpha_1\alpha_1)}{(\alpha_1\theta_1+x^3)(\alpha_2\theta_2+x^3)} - (\theta_1-\theta_2)$$

Thus, for$(\theta_1 > \theta_2 and \alpha_1 = \alpha_2)$ $or(\alpha_1 > \alpha_2 and \theta_1 = \theta_2)$; $\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

This shows the flexibility of the two-parameter Pranav distribution over Pranav and Ishita distribution.

2.4 Mean Deviation

The mean deviation about the mean and median are defined by

$$\delta_1(x) = \int_0^\infty |X-\mu| f(x)dx and \delta_2(x) = \int_0^\infty |X-M| f(x)dx \; .$$

respectively where$\mu = E(x)$ and M = median(X). The measures of $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the following relationships.

$$\delta_1(x) = \int_0^\mu (\mu-x)f(x)dx + \int_\mu^\infty (x-\mu)f(x)dx$$

$$= \mu F(\mu) - \int_0^\mu xf(x)dx - \mu[1-F(\mu)] + \int_\mu^\infty xf(x)dx$$

$$= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty xf(x)dx$$

$$= 2\mu F(\mu) - 2\int_0^\mu xf(x)dx \qquad 2.26$$

and

$$\delta_2(x) = \int_0^M (M-x)f(x)dx + \int_M^\infty (M-\mu)f(x)dx$$

$$= MF(M) - \int_0^M xf(x)dx - M[1-F(M)] + \int_M^\infty xf(x)dx$$

$$= \mu + 2\int_M^\infty xf(x)dx \; = \mu - 2\int_0^M xf(x)dx \qquad 2.27$$

Using pdf (2.1) and the mean of two-parameter Pranav distribution, it can be written as

$$\int_0^{\mu} xf(x)dx = \mu - \frac{[\alpha\theta^5\mu+\theta^4(\mu^4+\alpha)+4\theta^3\mu^3+12\theta^2\mu^2+24\theta\mu+24]}{\theta(\alpha\theta^4+6)}e^{-\theta x} \quad 2.28$$

$$\int_0^{M} xf(x)dx = \mu - \frac{[\alpha\theta^5 M+\theta^4(M^4+\alpha)+4\theta^3 M^3+12\theta^2 M^2+24\theta\mu+24]}{\theta(\alpha\theta^4+6)}e^{-\theta x} \quad 2.29$$

Using expression (2.17), (2.18), (2.19) and (2.20), the mean deviation about the mean $\delta_1(x)$ and the mean deviation about the median $\delta_2(x)$ of the two-parameter Pranav distribution are expressed as

$$\delta_1(x) = \frac{2(\alpha\theta^4+\theta^3\mu^3+6\theta^2\mu^2+18\theta\mu+24)e^{-\theta x}}{\theta(\alpha\theta^4+6)} \quad 2.30$$

$$\delta_2(x) = \frac{2[\alpha\theta^5 M+\theta^4(M^4+\alpha)+4\theta^3 M^3+12\theta^2 M^2+24\theta\mu+24]e^{-\theta x}}{\theta(\alpha\theta^4+6)} - \mu \quad 2.31$$

Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves and Bonferroni and Gini indices are given by Bonferroni (1930) have applications in almost every fields of knowledge including economics to study income and poverty of any state. It's relevance also in other fields like reliability, demography, insurance, and medicine. The Bonferroni and Lorenz curves are obtained as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty xf(x)dx\right] \quad 2.32$$

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{\mu}\left[\mu - \int_q^\infty xf(x)dx\right] \quad 2.33$$

Also, these can be expressed as

$$B(p) = \frac{1}{p\mu}\int_0^p F^{-1}(x)dx \quad 2.34$$

And $L(p) = \frac{1}{\mu}\int_0^p F^{-1}(x)dx$ 2.35

where $\mu = E(x)$ and $q = F^{-1}(p)$

The Bonferroni and Gini indices are expressed as

$$B = 1 - \int_0^1 B(p)dp \quad 2.36$$

$L = 1 - 2\int_0^1 B(p)dp$ respectively

Using the pdf of two-parameter Pranav distribution (2.1), we obtain

$$\int_q^\infty xf(x)dx = \frac{\{\alpha\theta^5 q+\theta^4(q^4+\alpha)+4\theta^3 q^3+12\theta^2 q^2+24\theta q+24\}e^{-\theta x}}{\theta(\alpha\theta^4+6)} \quad 2.37$$

Now using equation (7.7) in (7.1) and (7.2)

$$B(p) = \frac{1}{p}\left[1 - \frac{\{\alpha\theta^5 q+\theta^4(q^4+\alpha)+4\theta^3 q^3+12\theta^2 q^2+24\theta q+24\}e^{-\theta x}}{(\alpha\theta^4+6)}\right] \quad 2.38$$

$$L(p) = 1 - \frac{\{\alpha\theta^5 q+\theta^4(q^4+\alpha)+4\theta^3 q^3+12\theta^2 q^2+24\theta q+24\}e^{-\theta x}}{(\alpha\theta^4+6)} \quad 2.39$$

Now using equations (2.29) and (2.30), the Bonferroni and Gini indices of the distribution is thus obtained as

$$B = 1 - \frac{\{\alpha\theta^5 q+\theta^4(q^4+\alpha)+4\theta^3 q^3+12\theta^2 q^2+24\theta q+24\}e^{-\theta x}}{(\alpha\theta^4+6)} \quad 2.40$$

$$G = \frac{2\{\alpha\theta^5 q+\theta^4(q^4+\alpha)+4\theta^3 q^3+12\theta^2 q^2+24\theta q+24\}e^{-\theta x}}{(\alpha\theta^4+6)} \quad 2.41$$

Renyi Entropy Measure

A popular entropy measure is given by Renyi entropy. If X is a continuous random variable having probability density function(.), then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma} log\{f^\gamma(x)dx\}, \quad \gamma > 0 \text{ and } \gamma \neq 0 \quad 2.42$$

Thus, the Renyi entropy for two-parameter Pranav distribution (2.1) can be obtained as

$$T_R(\gamma) = \frac{1}{1-\gamma} log\left[\int_0^\infty \frac{\theta^{4\gamma}}{(\alpha\theta^4+6)^\gamma}(\alpha\theta + x^3)^\gamma e^{-\theta\gamma x}dx\right]$$

$$= \frac{1}{1-\gamma} log\left[\int_0^\infty \frac{\theta^{4\gamma}}{(\alpha\theta^4+6)^\gamma}\alpha\theta^\gamma\left(1+\frac{x^3}{\alpha\theta}\right)^\gamma e^{-\theta\gamma x}dx\right]$$

$$= \frac{1}{1-\gamma} log\left[\int_0^{\infty} \frac{\alpha\theta^{5\gamma}}{(\alpha\theta^4+6)^{\gamma}} \sum_{j=0}^{\infty} \binom{\gamma}{j} \left(\frac{x^3}{\alpha\theta}\right)^j e^{-\theta\gamma x} dx\right]$$

$$= \frac{1}{1-\gamma} log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\alpha\theta^{5\gamma}}{(\alpha\theta^4+6)^{\gamma}} \int_0^{\infty} e^{-\theta\gamma x} x^{3j} dx\right]$$

$$= \frac{1}{1-\gamma} log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\alpha\theta^{5\gamma-j}}{(\alpha\theta^4+6)^{\gamma}} \int_0^{\infty} e^{\theta\gamma x} x^{3j+1-1} dx\right]$$

$$= \frac{1}{1-\gamma} log\left[\sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\alpha\theta^{5\gamma-4\gamma-1}}{(\alpha\theta^4+6)^{\gamma}} \frac{\Gamma(3j+1)}{(\gamma)^{3j+1}}\right] \quad 2.43$$

Parameter Estimation

In this section, the estimations of parameters using the method of moments and the method of maximum likelihood have been discussed.

Method of Moments Estimates (MOME) of parameters

Method of moments can be calculated by equating population mean of two-parameter Pranav distribution to the sample mean, which is as follows.

MOME $\bar{\theta}$ of θ can be derived as

$$\bar{X} = \frac{\alpha\theta^4+24}{\theta(\alpha\theta^4+6)} = \mu_1' \quad 2.44$$

$$\theta\bar{X}(b+6) = b+24$$

Let $\alpha\theta^4 = b$

$$\hat{\theta} = \frac{b+24}{(b+6)\bar{X}} \quad 2.45$$

To obtain the estimate of α, and then substitute $\hat{\alpha} = \frac{b}{\theta^4}$ in equation 2.35, we have

$$\hat{\alpha} = \frac{b(b+6)^4\bar{X}^4}{(b+24)^4} \quad 2.46$$

Method of Maximum Likelihood Estimates

Let $(\boldsymbol{x_1, x_2, x_3, \ldots, x_n})$ be a random sample of size n from (2.1). The likelihood function L of two-parameter Pranav distribution is given by

$$\boldsymbol{L = \left(\frac{\theta^4}{\alpha\theta^4+6}\right)^n \prod_{i=1}^{n}\left(\alpha\theta + x_i^3\right)e^{-n\theta\overline{X}}} \qquad 2.47$$

and its log-likelihood function is thus obtained as

$$\boldsymbol{lnL = nln}\theta^4 - nln(\alpha\theta^4+6) + \boldsymbol{\sum_{i=1}^{n} ln\left(\alpha\theta + x_i^3\right) - n\theta\overline{X}} \qquad 2.48$$

The maximum likelihood estimates (MLE) $\boldsymbol{\hat{\theta}}$ of θ

$$\frac{\partial lnl}{\partial\theta} = \boldsymbol{\frac{4n\theta^3}{\theta^4} - \frac{4n\alpha\theta^3}{\alpha\theta^4+6} + \sum_{i=1}^{n}\frac{\alpha}{(\alpha\theta + x_i^3)} - n\overline{X} = 0}$$

$$\boldsymbol{= \frac{4n}{\theta} - \frac{4n\alpha\theta^3}{\alpha\theta^4+6} + \sum_{i=1}^{n}\frac{\alpha}{(\alpha\theta+x_i^3)} - n\overline{X} = 0} \qquad 2.49$$

where$\boldsymbol{\overline{x}}$ is the sample mean

$$\boldsymbol{\frac{\partial lnL}{\partial\alpha} = \frac{-n\theta^4}{(\alpha\theta^4+6)} + \sum_{i=1}^{n}\frac{\theta}{(\alpha\theta+x_i^3)} = 0} \qquad 2.50$$

The maximum likelihood estimates, $\boldsymbol{\hat{\theta}}$ and $\boldsymbol{\hat{\alpha}}$ of θ, and α are the solutions of the non-linear equation. The method of Fisher's scoring will be applied to solve these equations since these log-likelihood equations are difficult to express in closed forms.

$$\frac{\partial^2 lnL}{\partial\theta^2} = -\frac{4n}{\theta^2} - \frac{12n\alpha\theta^2}{\alpha\theta^4+6} + \frac{16n\alpha^2\theta^6}{(\alpha\theta^4+6)^2} - \sum_{i=1}^{n}\frac{\alpha^2}{(\alpha\theta+x_i^3)^2} \qquad 2.51$$

$$\frac{\partial^2 lnL}{\partial\alpha^2} = \frac{n\theta^8}{(\alpha\theta^4+6)^2} - \sum_{i=1}^{n}\frac{\theta^2}{(\alpha\theta+x_i^3)^2} \qquad 2.52$$

$$\frac{\partial^2 lnL}{\partial\theta\partial\alpha} = -\frac{24n\theta^3}{(\alpha\theta^4+6)^2} - \sum_{i=1}^{n}\frac{x_i^3}{(\alpha\theta+x_i^3)^2} \qquad 2.53$$

The following equation of $\hat{\theta}$ and $\hat{\alpha}$ can be solved as

$$\begin{bmatrix} \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL}{\partial\theta} \\ \frac{\partial lnL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

Where θ_0 and α_0 are initial values of θ and α respectively. Closes estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained by solving these equations iteratively.

Application and Goodness of fit

The two-parameter Pranav distribution can be applied to numerous lifetime data such as medical, biosciences, engineering and so many others. In this section, we present the application of the two-parameter Pranav to a real-life data set of hypertensive patients in which death is the event or failure. Its goodness fit was compared with Two-parameter Lindley, Two-parameter Akash, one parameter Pranav, Ishita, Akash, Sujatha, Shanker, Lindley, and Exponential distributions.

Data set: the data set represents the lifetime's data relating to times (in months from 1st January 2013 to 31st July 2018) of 105 patients who were diagnosed with hypertension and received at least one treatment-related to hypertension in the hospital where death is the event of interest.

The data are as follows: 45, 37, 14, 64, 67, 58, 67, 55, 64, 62, 9, 65, 65,43, 13, 8, 31, 30, 66, 9, 10, 31, 31, 31, 46, 37, 46, 44,45, 30, 26, 28, 45, 40, 47, 53, 47, 41, 39, 33, 38, 26, 22, 31, 46, 47, 66, 61, 54, 28, 9, 63, 56, 9, 49, 52, 58, 49, 53, 63, 16, 67, 61, 67, 28, 17, 31, 46, 52, 50, 30, 33, 13, 63, 54, 63, 56, 32, 33, 37, 7, 56, 1, 67, 38, 33, 22, 25, 30, 34, 53, 53, 41, 45, 59, 59, 60, 62, 14, 57, 56, 57, 40, 44, 63.

To compare these distributions, minimum, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) for the real data sets have been computed. The formulae for computing AIC and BIC are as follows:

$$AIC = -2lnl + 2k$$

$$BIC = -2lnl + klnn$$

Where K is the number of parameters and n is the sample size.

Results

The new proposed distribution was compared with Two-parameter Lindley, Two-parameter Akash, One parameter Pranav, Ishita, Sujatha, Akash, Shanker, Lindley, and Exponential distribution. These distributions were applied to hypertensive data to determine the most efficient distribution.

Table 1: *goodness of fit criteria*

Distributions	MLE's	Minimum	AIC	BIC
Two Parameter Pranav	$\theta = 0.09156147$ $\alpha = 83.6644763$	459.4631	922.9262	928.2341
Two Parameter Lindley	$\theta = 0.04431234$ $\alpha = 0.000004373$	473.1217	950.2435	955.8185
Two Parameter Akash	$\theta = 0.06994167$ $\alpha = 10.72072543$	463.7301	931.4603	937.0353
Pranav	$\theta = 0.09489049$	464.7007	931.4013	934.0553
Ishita	$\theta = 0.07115885$	465.2082	932.4164	935.0704
Sujatha	$\theta = 0.07023822$	464.639	931.278	933.9319
Akash	$\theta = 0.0710344$	464.6639	931.3277	933.9817
Shanker	$\theta = 0.04737547$	472.676	947.3519	950.0059
Lindley	$\theta = 0.0463843$	473.4971	948.9942	951.6481
Exponential	$\theta = 0.02371823$	497.8593	997.7186	1000.373

The best distribution corresponds to lower minimum, AIC and BIC. It can be easily seen from the above table that the two-parameter Pranav distribution is better than the two-parameter Akash, Two-parameter Lindley, Pranav, Ishita, Sujatha, Akash, Shanker, Lindley and exponential distributions for modeling lifetime data and thus the two-parameter Pranav distribution should be preferred to the other distributions mentioned in this work for modeling lifetime data-sets.

Discussion

A two-parameter Pranav distribution has been proposed which has its base from Pranav and Ishita distribution. Its mathematical and statistical properties which include its moments, coefficient of variation, skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, and stochastic ordering have been discussed. Further, expressions for Bonferroni and Lorenz curves and Renyi entropy measure of the proposed distribution was also derived. The method of moments and maximum likelihood estimation for estimating the distribution parameter has been discussed as well. An example of a real-life data set has been presented to show the application and goodness of fit of the two-parameter Pranav over Two-parameter Lindley, Two-parameter Akash, Pranav, Ishita, Sujatha, Akash, Shanker, Lindley, and exponential distributions.

Conclusion

The result of the analysis indicates that the Two-parameter Pranav has the lowest value of AIC and BIC. Remember the lower the AIC and BIC value, the better the distribution. Therefore, the values of minimum, AIC, and BIC of the distributions shows the superiority of the two-parameter Pranav over the other distributions. Hence two-parameter Pranav can be seen as an important distribution in modeling lifetime data.

Conflict of interest

None

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